A1 Dynamical Astronomy

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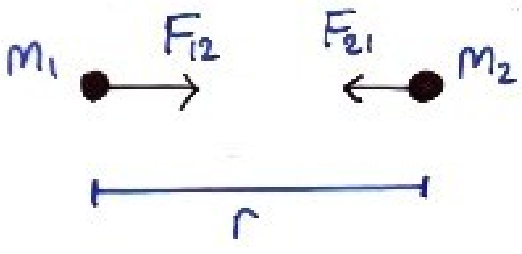
We will :

- Look at the motions of the planets, stars, galaxies + spacecraft within the universe
- Analyse how they interact under gravity

- To do this, we need to consider the general rules ('laws') of motion for bodies and the 'Universal Law of Gravitation', all deduced by Isaac Newton in the 17th century.

Gravity

"Any two point masses feel a mutual attractive force proportional to the product of the masses and inversely proportional to the square of their separation." This is The Universal Law of Gravitation



$$F_{12} = F_{21} = G\,\frac{m_1 m_2}{r^2}$$

'Constant of Gravitation'

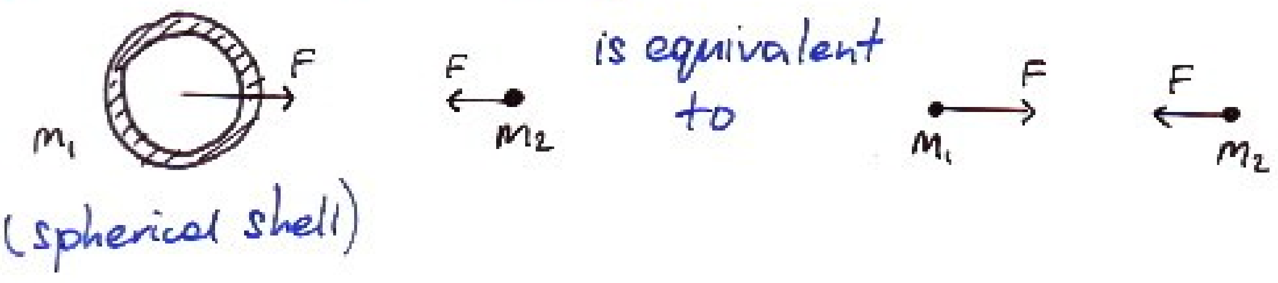
In SI units $G \simeq 6.67 \times 10^{-11}\ Nm^2 kg^{-2}$

Note:

- Both masses feel the same force ($F_{12} = F_{21}$)
- The force is always attractive, so can't be cancelled out like, eg, electrical forces can.
- Gravity is the only long-range force that is purely attractive, & therefore dominates on astronomical scales (though is hardly noticed between everyday objects).

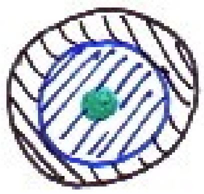
What about masses that aren't points?

Generally mathematically awkward, but spherically symmetric masses are easy:



ie a spherical shell of mass, seen from outside, attracts as if all its mass was concentrated at the centre (no proof here)

This is therefore also true of objects that can be constructed out of concentric spherical shells, eg planets:

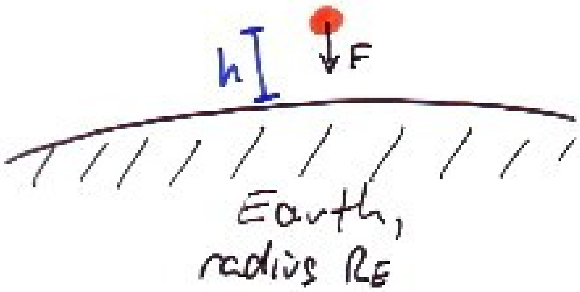


total mass M

is equivalent to • M

[If you are below the surface, shells above you exert no net gravitational force. Again, no proof here !]

- so the gravitational force of attraction between an orange, mass M_o, and the Earth, mass M_E, is simply



$$F = \frac{G M_o M_E}{(R_E + h)^2}$$

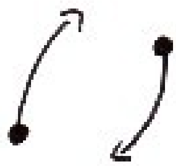
* We will assume that we can model stars, planets and spacecraft as point-masses in this way *

Motion under gravity

The resulting motion can be very simple if both bodies start at rest:

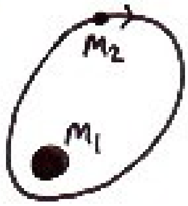
bodies: stick together (if solid)
coalesce (if fluid)

- But if either body has a small velocity not directed along the line joining their centres, they might miss:



A property of any inverse-square law of attraction is that the resulting motion of the two bodies form closed, repeating orbits

$m_1 = m_2$



$m_1 \gg m_2$

$m_1 \gg m_2$
and initial speed of m_2 carefully chosen

- Much of this course is concerned with analysing the properties of these (generally elliptical) orbits.

Many-body systems

If more than two bodies are involved in the system it is called a many-body or N-body system.

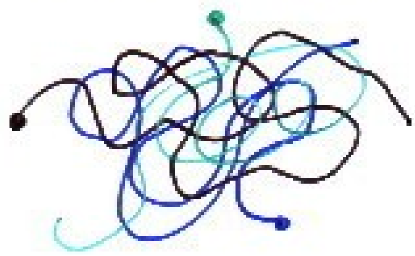
The rules are simple:

1) Work out the total force on each mass

2) Move each mass slightly in accordance with Newton's laws of motion (see later)

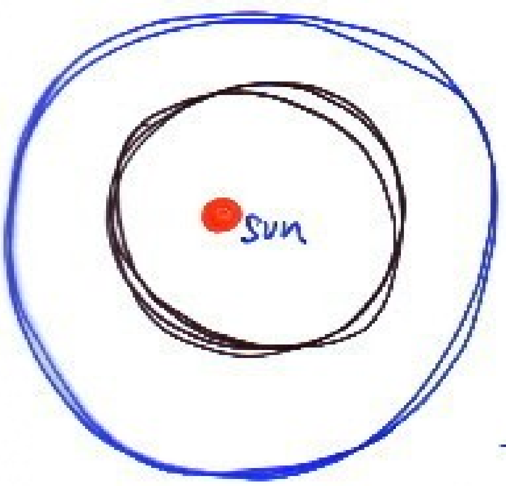
3) Go back to 1) [The forces will have changed!]

but in general the motion is VERY complicated:



- can only be analysed on a computer.

- We will stick to 2-body problems!
- Note the motions of the planets are nearly-2-body:



The orbits of the planets are slighty perturbed by their attraction to each other.

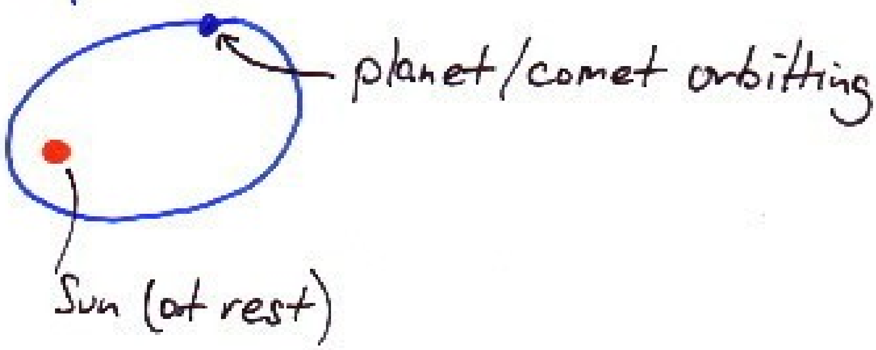
- But the Sun dominates, so we will ignore these perturbations.

1-body systems

Often in astronomy we are concerned with the interaction of two masses where one is much, much, greater than the other:

eg Sun - Earth system (or any Sun-planet system)
Earth - Space Station system
[but not really the Earth-Moon system]

- Often called a 1-body system, as only 1 body moves significantly.



Johannes Kepler (1571-1630) famously derived three rules that the planets appear to obey while orbitting the Sun :

Kepler's Laws of Planetary Motion

(K1) The orbit of each planet is an ellipse, with the Sun at one focus

(K2) For any planet, the line joining the planet to the sun sweeps out equal areas in equal times

(K3) The cubes of the semi-major axes of the planetary orbits are proportional to the squares of the planets orbital periods $\left(\frac{a^3}{T^2} = \text{const}\right)$

We can understand these in terms of the properties of ellipses and the predictions of Newtonian Dynamics ...

Newton's Laws of Motion

- How and why things move:

NI: "Every body continues in its state of rest or uniform motion in a straight line until acted on by an external force."

[planets don't move in straight lines ⇒ forces acting]

NII: "The rate of change of momentum of a body is proportional to the applied force, and is in the direction of the force."

$$\text{momentum} = \text{mass} \times \text{velocity}$$

$$\underline{p} = m\underline{v} \qquad (\underline{p} + \underline{v} \text{ are vectors})$$

$$\underline{F} = \frac{d}{dt}(m\underline{v}) = \underbrace{m\frac{d\underline{v}}{dt} = \text{mass} \times \text{acceleration}}_{\text{if } m \text{ is constant}}$$

Note that if the total force acting on a system is zero ($F=0$), momentum is constant.

⇒ the conservation of linear momentum

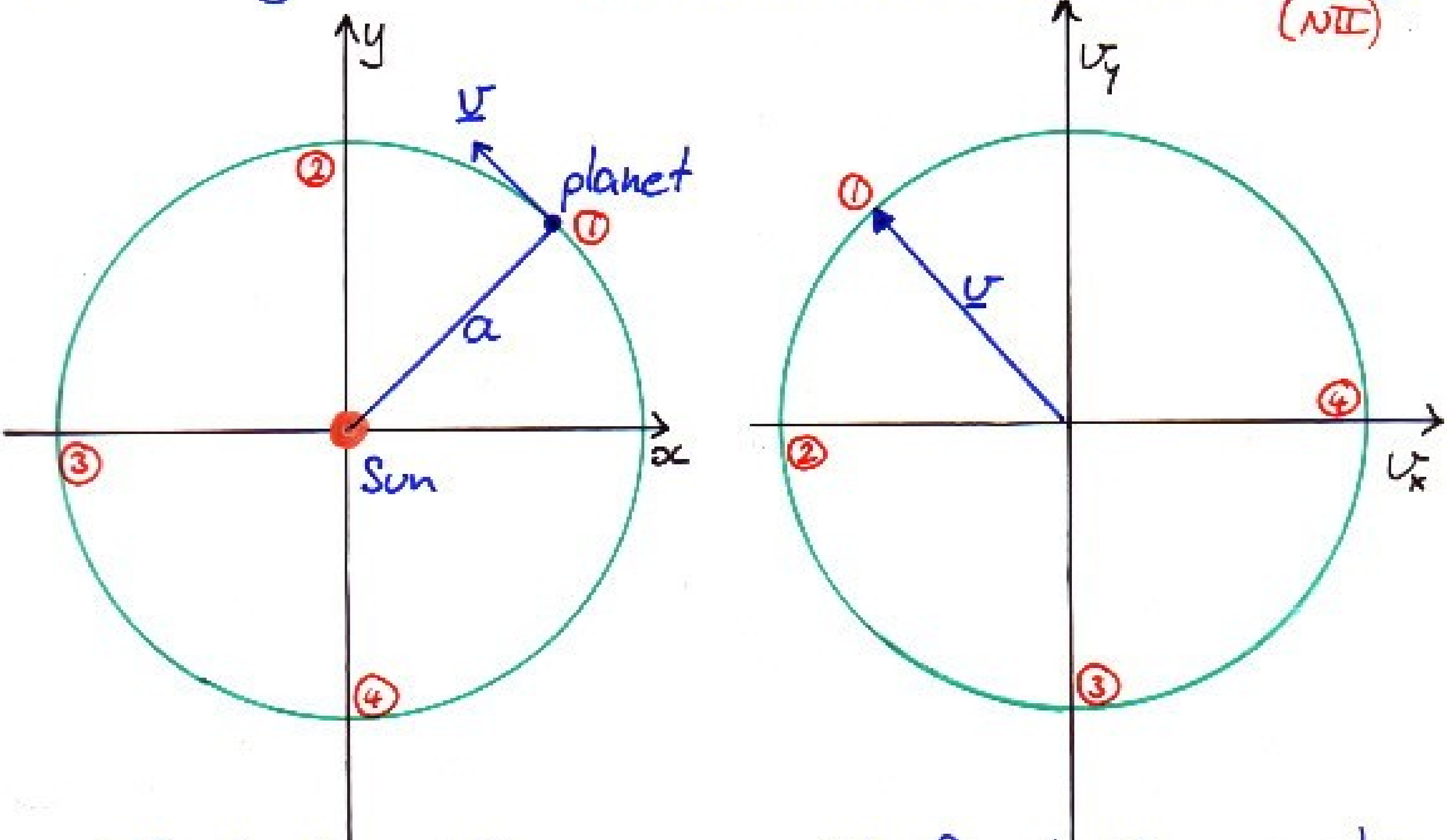
NIII: "To every action, there is an equal and opposite reaction"

⇒ Force of the Sun on a planet equals (−) the force of the planet on the Sun.

Force of rocket on exhaust equals (−) the force of the exhaust on the rocket etc...

Circular Motion

A planet travelling in a circular orbit <u>must</u> be feeling a force (NI), and must be accelerating (NII)



plot of planet's position over time

plot of velocity over time

- Time taken for planet to go around once is its orbital period, T (sometimes 'P')

speed $= v$; distance in time $T = 2\pi a$

$$\Rightarrow \quad v = \frac{2\pi a}{T} \quad \text{in circular motion.}$$

Can define angular velocity, $\omega = \frac{2\pi}{T}$

'omega'

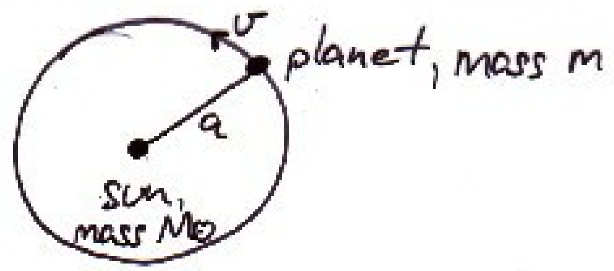
so $v = a\omega$

- In time T, $\underline{v}$ changes by $2\pi v$ (circumf. of r.h. circle)

so acceleration $= \frac{2\pi v}{T}$

$$\left.\begin{aligned} &= v\omega \\ &= \omega^2 a \\ &= \frac{v^2}{a} \end{aligned}\right\} \text{equivalent}$$

- This is the 'centripetal acceleration' needed for circular motion.
- The centripetal force that generates it in planetary motion is gravity.
- Both the force + the acceleration are directed towards the Sun



So if $\quad$ Force = mass × acceleration

we have for the planet:

$$\frac{G M_\odot m}{a^2} = m\,\omega^2 a = m \frac{4\pi^2}{T^2} a$$

$$\Rightarrow \quad \boxed{a^3 = \frac{G M_\odot}{4\pi^2} T^2}$$

So we have proved Kepler's third law $\left(\frac{a^3}{T^2} = \text{const}\right)$ for the case of a circular orbit [we have not yet proved it for more general elliptical orbits]

- Note that the constant $\frac{G M_\odot}{4\pi^2}$ is the same for all planets orbiting the same star. Its numerical value will depend on the units we choose for a and T.

If we measure a in AU and T in sidereal years

then $\quad a^3 = T^2 \quad$ [check for Earth: $1^3 = 1^2$]

These are the natural units for the solar system.
[1 AU = 1.496×10^{11} m ; 1 year = 3.1558×10^7 s]

Geostationary Orbits

A satellite is 'geostationary'
if it orbits the Earth in the equatorial plane with a period of 1 sidereal day (= spin period of Earth, $23^h 56^m$, so is 'geosynchronous')

This satellite will remain above the same point on the Earth's equator as it orbits.

To find orbital radius, use $a^3 = \frac{GM}{4\pi^2} T^2$ — of Earth

- Acceleration due to gravity on the Earth's surface:

$$g = \frac{\text{grav. force on } m}{m} = \frac{GM}{R_E^2}$$

radius of Earth

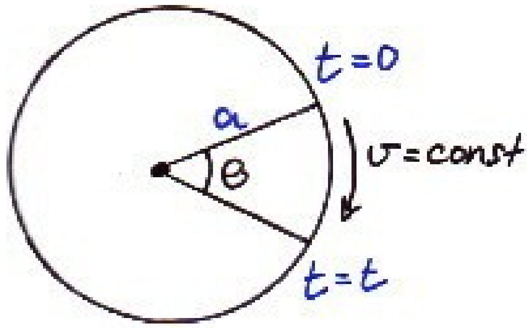
so
$$a^3 = \frac{g R_E^2}{4\pi^2} T^2$$

$$\frac{a}{R_E} = \left(\frac{g T^2}{4\pi^2 R_E}\right)^{1/3}$$

$$\left.\begin{array}{l} g = 9.8\,\mathrm{m\,s^{-2}} \\ T = 23^h 56^m = 8.62\times10^4\,\mathrm{s} \\ R_E = 6.38\times10^6\,\mathrm{m} \end{array}\right\} \Rightarrow \frac{a}{R_E} = 6.61, \text{ or } a = 4.22\times10^7\,\mathrm{m}$$

KII and circular orbits

Kepler's 2^{nd} law is clearly true for circular orbits:



$$\text{area swept out} = \frac{\theta}{2\pi} \cdot \pi a^2 = \frac{1}{2}\theta a^2$$

$$\text{angle } \theta = \frac{vt}{a}$$

$$\Rightarrow \quad \text{area swept out} = \frac{1}{2}\frac{vt}{a}a^2 = \frac{1}{2}vat$$

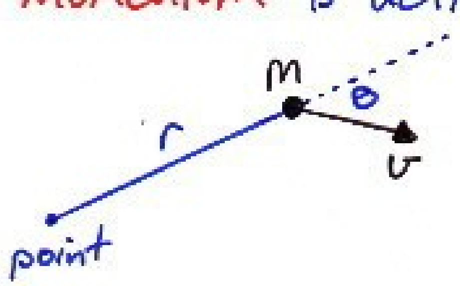
$$\text{rate of sweeping out area} = \frac{\text{area}}{\text{time}} = \frac{1}{2}va = \text{constant}$$

- This is a specific example of a general rule derivable from Newton's laws of motion:

Angular Momentum, L

We have met linear momentum $\underline{p} = m\underline{v}$, changed by linear forces.

Angular momentum is defined about a point:



$$L = mvr\sin\theta$$

and is changed by 'twisting' (torque) forces

For circular motion

$$L = mva$$

so rate of sweeping out area $= \frac{1}{2} va = \frac{L}{2m}$ = constant.

- Turns out that rate of sweeping out area $\propto L$ in the case of elliptical orbits too.

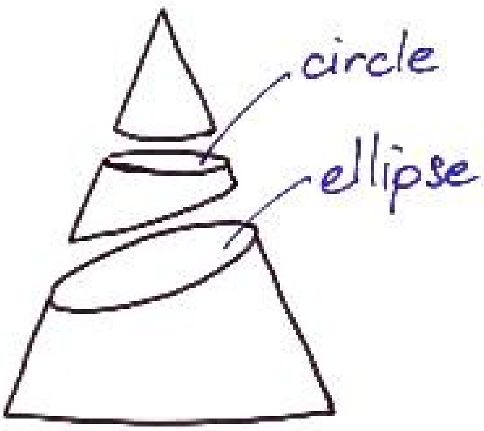
Angular momentum is conserved in orbits because the force of gravity acts along a line joining the centres of the bodies (if spherically symmetric), and cannot exert a torque on them.

$\Rightarrow$ KII is a consequence of the conservation of angular momentum in orbital motion.

To understand the general forms of Kepler's laws, we need to look at elliptical orbits..

Elliptical Orbits

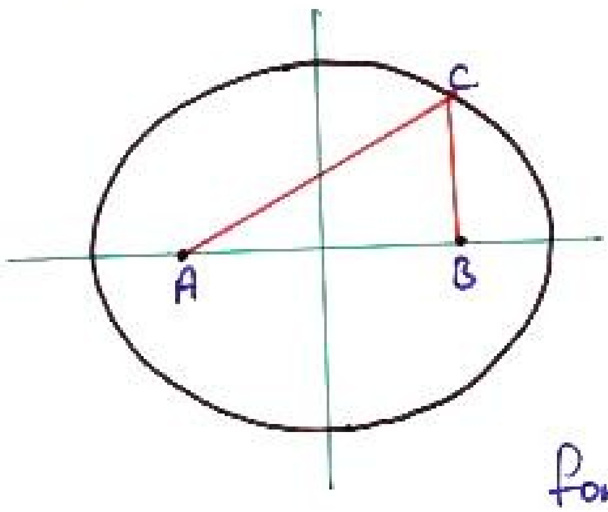
An ellipse is a conic section – a section of a right circular cone:



a circle is a special case of the ellipse

A cut at an angle parallel to the slope of the cone gives a parabola, still larger angles a hyperbola.

All these curves are valid 'orbital' trajectories, but only elliptical orbits are closed.



An ellipse has two foci, A & B

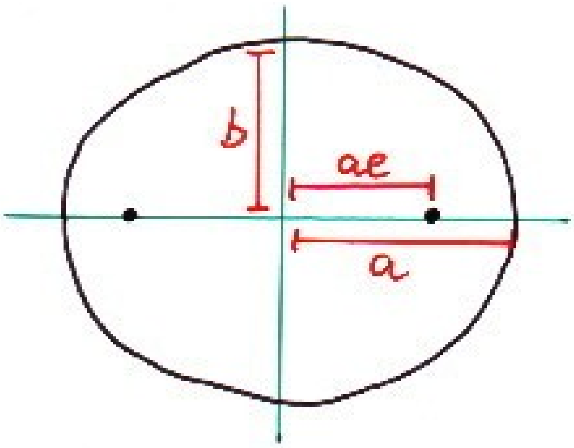
Property #1:

$AC + BC = \text{constant}$

for all C on the ellipse

$\Rightarrow$ can draw one with 2 pins & a length of string.

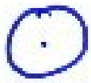
Ellipse dimensions :



a = 'semi-major axis'

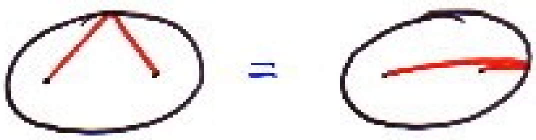
b = 'semi-minor axis'

e = 'eccentricity'

- If $e = 0$ we have a circle 

  If $e \lesssim 1$ we have a long, thin orbit 

  [$e = 1$ corresponds to a parabola ; $e > 1$ to a hyperbola]

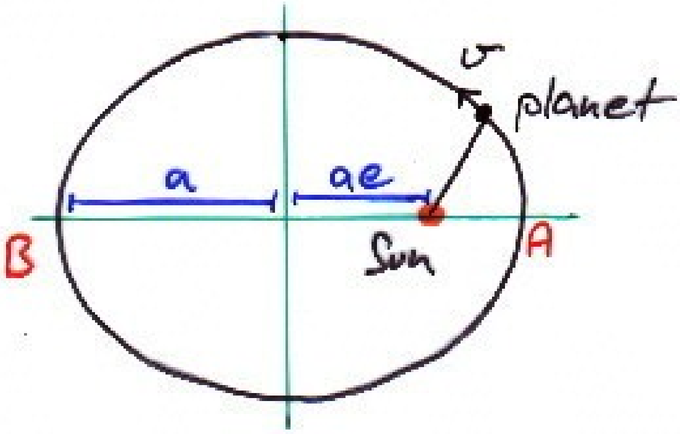
- From property #1 

  ie $$2\left(b^2 + a^2e^2\right)^{1/2} = (ae + a) + (a - ae)$$

  $$b^2 + a^2e^2 = a^2$$

  $$b^2 = a^2(1 - e^2)$$

- KI states that planets orbit in ellipses with the Sun at one focus



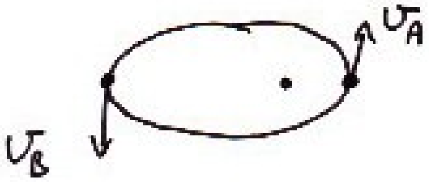
Two important points are A + B, where the planet is nearest + furthest from the Sun

A = 'perihelion' , $a(1-e)$ from sun

B = 'aphelion' , $a(1+e)$ from Sun

[For orbits around Earth, terms are 'perigee' + 'apogee', 'periastron' and 'apastron' around stars, 'periapsis' and 'apoapsis' generally]

- KII ⇒ we can equate angular momentum at A + B



planet mass m

$$m v_A \, a(1-e) = m v_B \, a(1+e)$$

$$\frac{v_A}{v_B} = \frac{1+e}{1-e}$$

– a useful result (see later). Clearly $v_A > v_B$

Example Comet Encke has a perihelion distance of 0·339 AU and an orbital eccentricity of 0·847. Calculate:

i) Its orbital dimensions, a & b

ii) Its orbital period

iii) the ratio of its maximum & minimum speeds.

Answer:

i) Perihelion distance, $r_p = a(1-e)$

$$\Rightarrow \qquad a = \frac{r_p}{1-e} = \underline{2.216\ \text{AU}}$$

Also, $b = a(1-e^2)^{1/2} = 0.532a = \underline{1.178\ \text{AU}}$

ii) In natural units, $a^3 = T^2$

$$\Rightarrow \text{period} = a^{3/2}\ \text{years} = \underline{3.30\ \text{years}}$$

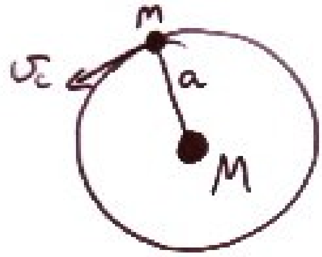
[remember, for this to work a must be in AU]

iii) Maximum speed happens at perihelion
minimum " " " aphelion

$$\Rightarrow \qquad \frac{v_{max}}{v_{min}} = \frac{1+e}{1-e} = \underline{12.07}$$

Response to an impulse

To get a circular orbit, the speed has to be just right:



mass × accel^{n} = Force

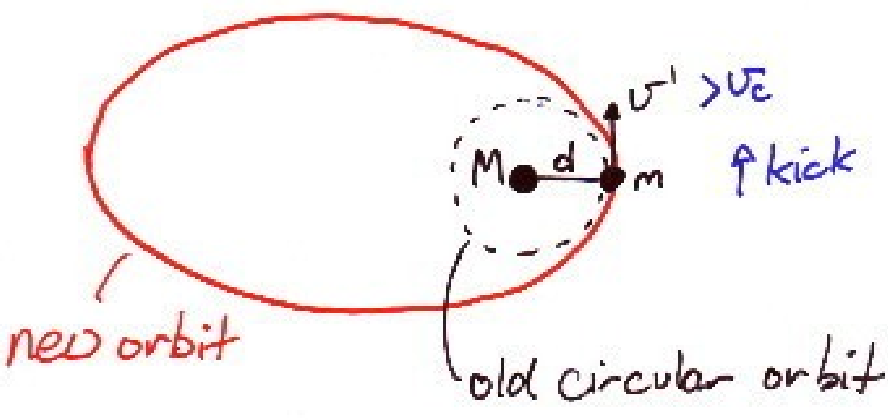
$$\frac{m v_c^2}{a} = \frac{GMm}{a^2}$$

$$\Rightarrow \quad v_c = \sqrt{\frac{GM}{a}}$$

What happens if we give the mass m a kick? (perhaps a rocket briefly firing, or the impact of an asteroid).

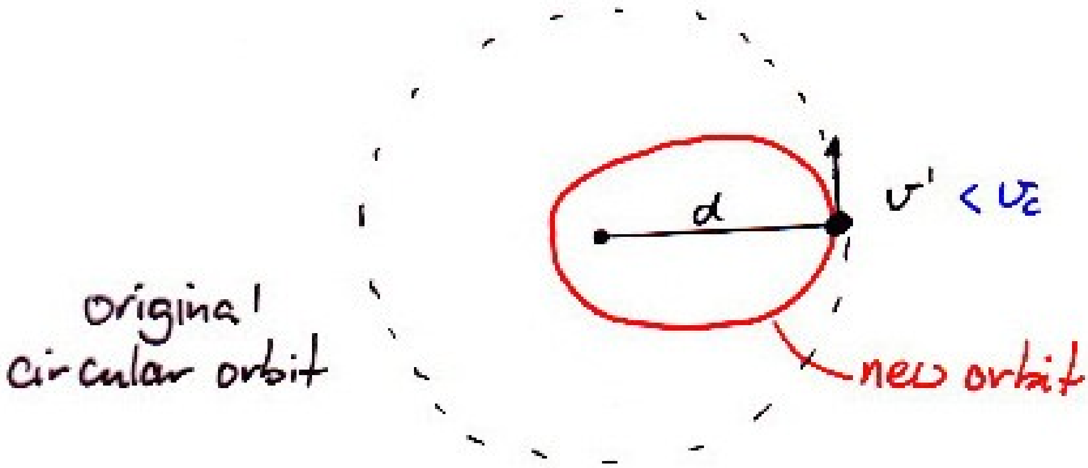
Case #1: kick in direction of motion

- m will speed up, so the orbit becomes elliptical rather than circular:



The distance d = radius of circular orbit
= periastron of new orbit

Case #2: kick in opposite direction to motion



The new velocity v' is now $< v_c$, but still perpendicular to d at the moment of the kick.

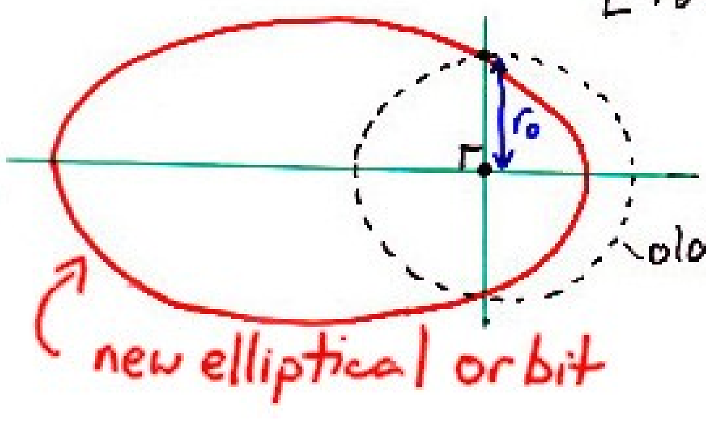
d now becomes the apastron distance of the new orbit.

Case #3: radial kicks

These apply no torque about the focus, so cannot change the angular momentum, L, but will change the velocity.

The semi latus rectum of the orbit, r_0, depends only on L, and is therefore preserved:

[$r_0 = L^2/GMm^2$ - no proof here]



Notes on kicks

- We have only considered 3 special cases
- Radial kicks ('in' or 'out') always increase the kinetic energy ⇒ larger orbits (ie greater 'a')
- Orbital manoeuvres can be counter-intuitive,

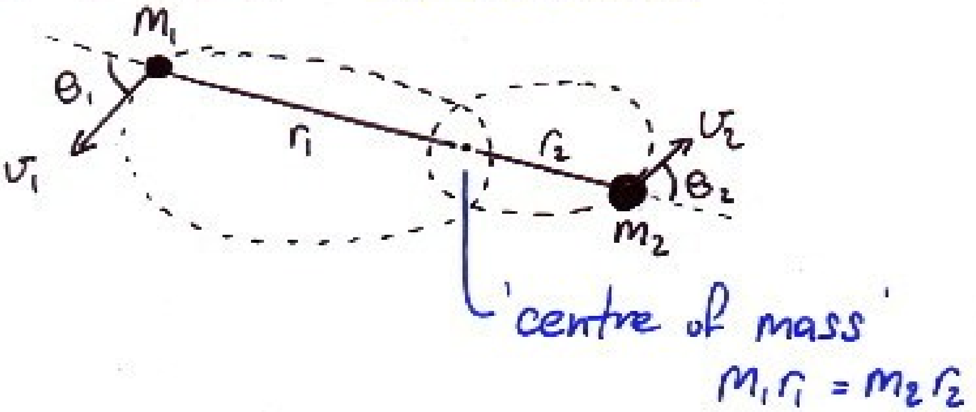
  eg, to descend to Earth, the space shuttle fires its engines to kick it in the opposite direction to its travel (see case #2), not to kick it radially towards the Earth. (That would be case #3 - it would end up mostly further away!)
- An orbit will change shape too if the central object change its mass, M. (because v_c depends on M)
- To get a fuller analysis we need to consider the energy of an orbit ...

Conserved Quantities

Orbital analysis is greatly helped by considering numbers defined by the motion that can be calculated at any time and always remain the same.

These are 'conserved quantities'.

Linear momentum is conserved:



Two stars $m_1 + m_2$ orbiting their common centre of mass (assumed at rest)

$$m_1 v_1 = m_2 v_2 \quad \text{at all times}$$

and the velocities have opposite directions.

Angular momentum is conserved too:

$$L_{total} = m_1 r_1 v_1 \sin\theta_1 + m_2 r_2 v_2 \sin\theta_2$$

$$= \text{constant at all times} \quad \text{(if no external torques act)}$$

These are fundamental laws of physics

Another fundamental law of physics is that total Energy is conserved.

- What is the 'total energy' of a planet or satellite?

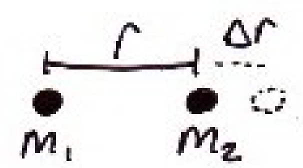
Energy

For most orbital motion we need only consider the kinetic energy and gravitational potential energy:
K.E. P.E. (or V)

- Mass m moving at speed v has kinetic energy:

$$K.E. = \frac{1}{2} m v^2$$

- The gravitational potential energy of two masses, m_1 and m_2 is (minus) the work done in separating them an infinite distance:



work done to move m_2 from r to $r + \Delta r$:

$$\Delta wd = \underbrace{\text{force}}_{\sim \text{constant over distance } \Delta r} \times \text{distance} = \frac{G m_1 m_2}{r^2} \Delta r$$

work done to separate them to infinity:

$$\text{total work done} = \int_r^\infty \frac{Gm_1m_2}{r^2}\,dr$$

$$= \frac{Gm_1m_2}{r}$$

$\Rightarrow$ gravitational PE $(= V) = -\dfrac{Gm_1m_2}{r}$ at separation r

Note:
- V is always negative ($V = 0$ at $r \to \infty$)
- it's negative because energy is given up when masses are brought together.

So for the 1-body problem

$$\boxed{\text{total energy } E = \frac{1}{2}mv^2 - \frac{GMm}{r} = \text{constant}}$$

Virial theorem

Take a circular orbit: $E = \frac{1}{2}mv^2 - \frac{GMm}{a} = \text{const.}$

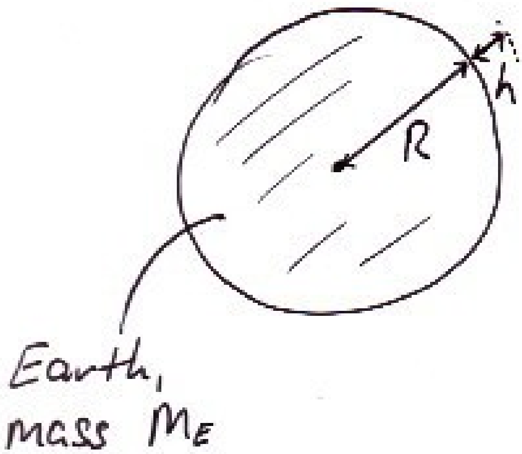
But NII $\Rightarrow$ $\frac{mv^2}{a} = \frac{GMm}{a^2}$, ie $mv^2 = \frac{GMm}{a}$

Hence $E = \frac{1}{2}\frac{GMm}{a} - \frac{GMm}{a} = -\frac{GMm}{2a} = \frac{P.E}{2}$

so $E = \frac{1}{2} PE$

$KE = -\frac{1}{2} PE$ – the 'virial theorem' for circular orbits.

Potential energy close to the Earth's surface



At surface $PE = -\dfrac{G M_E m}{R}$

For a small change in distance ΔR:

$$\Delta P.E. = \frac{G M_E m}{R^2} \Delta R$$

but $$mg = \frac{G M_E m}{R^2}$$

so $$\Delta PE = mg\Delta R = mgh \text{ (as before)}$$

Escape velocity

$$E = \frac{1}{2} m v^2 - \frac{GMm}{r}$$

If $E > 0$, then m escapes (ie solutions for v exist as $r \to \infty$)

- if m just escapes, $v = 0$ at $r = \infty$, ie $E = 0$

The escape velocity at r needed to do this is set by

$$\frac{1}{2} m v_{esc}^2 - \frac{GMm}{r} = 0$$

ie $$\boxed{v_{esc} = \left(\frac{2GM}{r}\right)^{1/2}}$$

Note:

- This is the ballistic velocity required – no power is being applied [you can escape at any speed with engines!]
- The direction of v is not important, as long as you don't crash into the central body!

Example: Escape velocity from Earth's surface

$$v = \left(\frac{2GM_E}{R_E}\right)^{1/2}$$

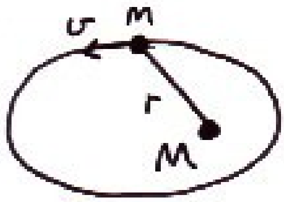
but $$mg = \frac{GM_E m}{R_E^2} \Rightarrow GM_E = gR_E^2$$

so $$v = (2gR_E)^{1/2}$$

$g = 9.8 \text{ m s}^{-2}$
$R_E = 6.38 \times 10^6 \text{ m}$

$$\Rightarrow \underline{\underline{v = 11.2 \text{ km s}^{-1}}}$$

Velocity in a general elliptical orbit



What is $v(r)$?
(the speed when r from focus)

Conservation of energy gives:

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}$$

or $$v^2 = \frac{2E}{m} + \frac{2GM}{r}$$

write as $$v^2 = GM\Big(\frac{2}{r} + \underbrace{\frac{2E}{GMm}}_{\text{a constant, } C}\Big)$$

so $$v(r) = \left[GM\left(\frac{2}{r} + C\right)\right]^{1/2}$$

- We can express the constant C in terms of orbital dimensions (rather than energy) by conserving angular momentum:

let v_1 = speed at perihelion, when $r = a(1-e)$
v_2 = " " aphelion, " $r = a(1+e)$

conservation of angular momentum $\Rightarrow$

$$v_1 a(1-e) = v_2 a(1+e) \qquad (1)$$

but also:

$$v_1^2 = GM\left[\frac{2}{a(1-e)} + C\right] \quad ; \quad v_2^2 = GM\left[\frac{2}{a(1+e)} + C\right] \qquad ②$$

from ②:

$$\frac{v_1^2}{v_2^2} = \frac{\frac{2}{a(1-e)} + C}{\frac{2}{a(1+e)} + C} = \frac{(1+e)}{(1-e)} \frac{[2 + Ca(1-e)]}{[2 + Ca(1+e)]}$$

but from ① we know that $\frac{v_1}{v_2} = \frac{1+e}{1-e}$

so $$\left(\frac{1+e}{1-e}\right)^2 = \left(\frac{1+e}{1-e}\right)\left[\frac{2 + Ca(1-e)}{2 + Ca(1+e)}\right]$$

or $$\frac{1+e}{1-e} = \frac{2 + Ca(1-e)}{2 + Ca(1+e)}$$

multiplying out:

$$2 + 2e + Ca(1 + 2e + e^2) = 2 - 2e + Ca(1 - 2e + e^2)$$

$$2e + 2Cae = -2e - 2Cae$$

$$4e = -4Cae$$

$$C = -\frac{1}{a}$$

so $$\boxed{v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)}$$

Also, C equals $\frac{2E}{GMm}$ by definition, so

$$a = -\frac{GMm}{2E}$$ [remember E is negative]

ie the semimajor axis of the orbit depends only on the total energy E (given M + m)

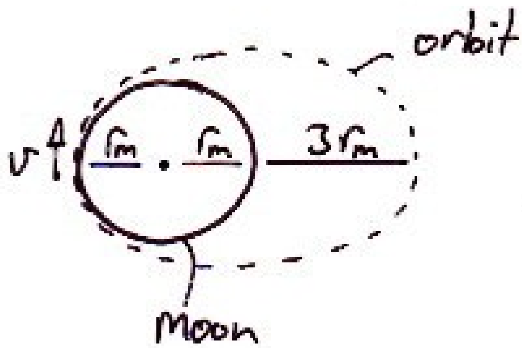
[likewise the period, from K III]

Example:

A bullet is fired horizontally on the moon, and goes into orbit. Its maximum altitude above the lunar surface is 3 lunar radii.

- Calculate the speed of the bullet when fired, and the time it takes to reach its maximum altitude.

Ans:



- Bullet fired horizontally ⇒ initial point must be either the peri- or ap-lunar point.

Max altitude > r_m ⇒ Must initially be the perilunar point.

From the diagram: $r_m + r_m + 3r_m = 2a$
(where a = semi-major axis)

$$\Rightarrow \quad a = \frac{5}{2} r_m$$

mass of Moon

Using $$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

at initial point $v^2 = GM\left(\frac{2}{r_m} - \frac{2}{5r_m}\right) = \frac{8}{5}\frac{GM}{r_m}$

$$v = \left(\frac{8}{5}\frac{GM}{r_m}\right)^{1/2}$$

$G = 6.67 \times 10^{-11}\ m^3\,kg^{-1}\,s^{-2}$
$M = 7.35 \times 10^{22}\ kg$
$r_m = 1.74 \times 10^{6}\ m$

$$\Rightarrow \quad v = 2123\ ms^{-1}$$

- Time to max altitude = half orbital period.

Orbital period^2 $= T^2 = \dfrac{4\pi^2 a^3}{GM}$ (K III)

$$\Rightarrow \quad T^2 = \frac{4\pi^2 (4.35 \times 10^6)^3}{(6.67 \times 10^{-11}) \times (7.35 \times 10^{22})}$$

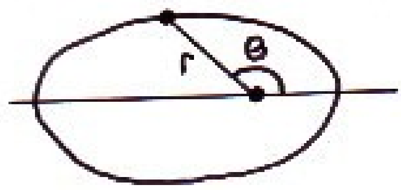
$$\Rightarrow \quad T = 25{,}746\,s \simeq 7.15 \text{ hours}$$

So time to max altitude $\simeq$ 3½ hours

(but watch out for its return!)

Equation of the Ellipse

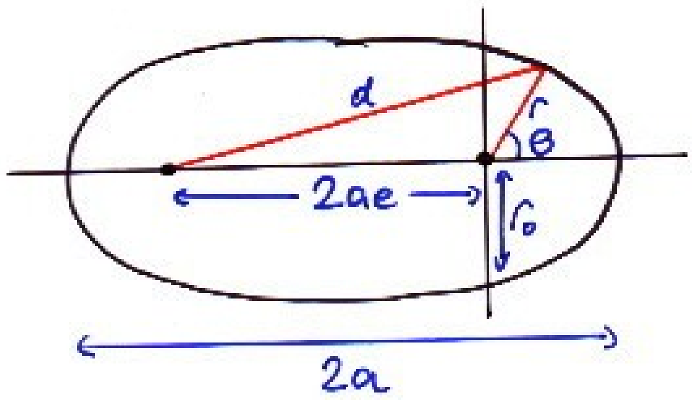
We have seen it is convenient to consider the ellipse in polar coordinates, r and θ.



The origin (of r) is the focus of the ellipse

θ is called the true anomaly of the orbiting body, and r its radius vector.

- We can determine $r(\theta)$ from the 'string' definition of the ellipse :



When $\theta = 0$, string length $= ae + a + (a - ae)$
$= 2a$

So quite generally, $d + r = 2a$

By the cosine rule

$$d^2 = 4a^2e^2 + r^2 - 4aer\cos(\pi - \theta)$$

ie $4a^2 - 4ar + r^2 = 4a^2e^2 + r^2 + 4aer\cos\theta$

$$4a^2(1-e^2) = 4aer\cos\theta + 4ar$$

$$a(1-e^2) = r(1+e\cos\theta)$$

when $\theta = 90°$, $r = r_0$ (the semi-latus rectum)

$$\Rightarrow \quad \boxed{\begin{aligned} r_0 &= a(1-e^2) \\ r &= \frac{r_0}{1+e\cos\theta} \end{aligned}}$$

[see kI proof]

Clearly, for a circle: $e = 0 \Rightarrow r_0 = a = r$

Relationship between r_0 and L

Take a body in orbit with semi-major axis a, and eccentricity e.

At perihelion, $r_p = a(1-e)$ [eg, from $\theta = 0$ above]

Its angular momentum, $L = m v_p r_p$ (as before)

but $v_p^2 = GM\left(\frac{2}{r_p} - \frac{1}{a}\right)$

so $L^2 = m^2 r_p^2 GM\left(\frac{2}{r_p} - \frac{1}{a}\right)$

ie $$L^2 = GMm^2\left(2a(1-e) - a(1-e)^2\right)$$
$$= GMm^2 a(1-e^2)$$

but $a(1-e^2) = r_0$, so

$$\boxed{L^2 = GMm^2 r_0}$$

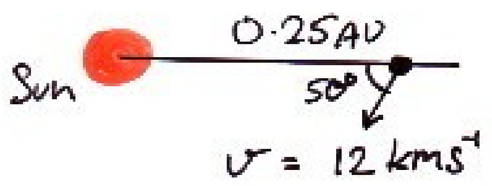
This explains why a radial kick will throw a body into a new orbit with the <u>same</u> r_0 length.
Radial kicks do not affect L, so they don't change r_0 either.

[note that the angular momentum per unit mass, L/m is sometimes written 'h', so that $h^2 = GMr_0$.]

<u>Example</u>

An asteroid is seen travelling at 12 km s^{-1}, making an angle of 50° with a line to the Sun when 0.25AU from the Sun.

Will it crash into the Sun?



We know v and r at the same time, so we can get a (the semi-major axis of the orbit)

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

$$\Rightarrow \quad \frac{1}{a} = \frac{2}{r} - \frac{v^2}{GM}$$

$1\,AU = 1.49 \times 10^{11}\,m$

$GM_\odot = 1.33 \times 10^{20}\,m^3 s^{-2}$

$$\Rightarrow \quad a = 1.9 \times 10^{10}\,m \quad (= 0.13\,AU)$$

The asteroid will impact on the sun if the perihelion distance, r_p, is less than the radius of the sun, $R_\odot$.

$$r_p = a(1-e)$$

so we need e ...

Consider angular momentum:

$$L^2 = GMm^2 r_0 = GMm^2 a(1-e^2)$$

by definition $L = mvr\sin\theta$

$$\text{so} \quad m^2 v^2 r^2 \sin^2\theta = GMm^2 a(1-e^2)$$

$$1 - e^2 = \frac{(vr\sin\theta)^2}{GMa}$$

$\theta = 50°$

$$\Rightarrow \quad e = 0.977$$

We are seeing the asteroid close to aphelion in a highly elliptical orbit.

Perihelion distance $= a(1-e)$

$= 4.4 \times 10^8 \text{ m}$

But $R_\odot = 7 \times 10^8 \text{ m}$, so the asteroid hits the sun

Speed of impact $= \left[GM\left(\frac{2}{R_\odot} - \frac{1}{a}\right)\right]^{1/2}$

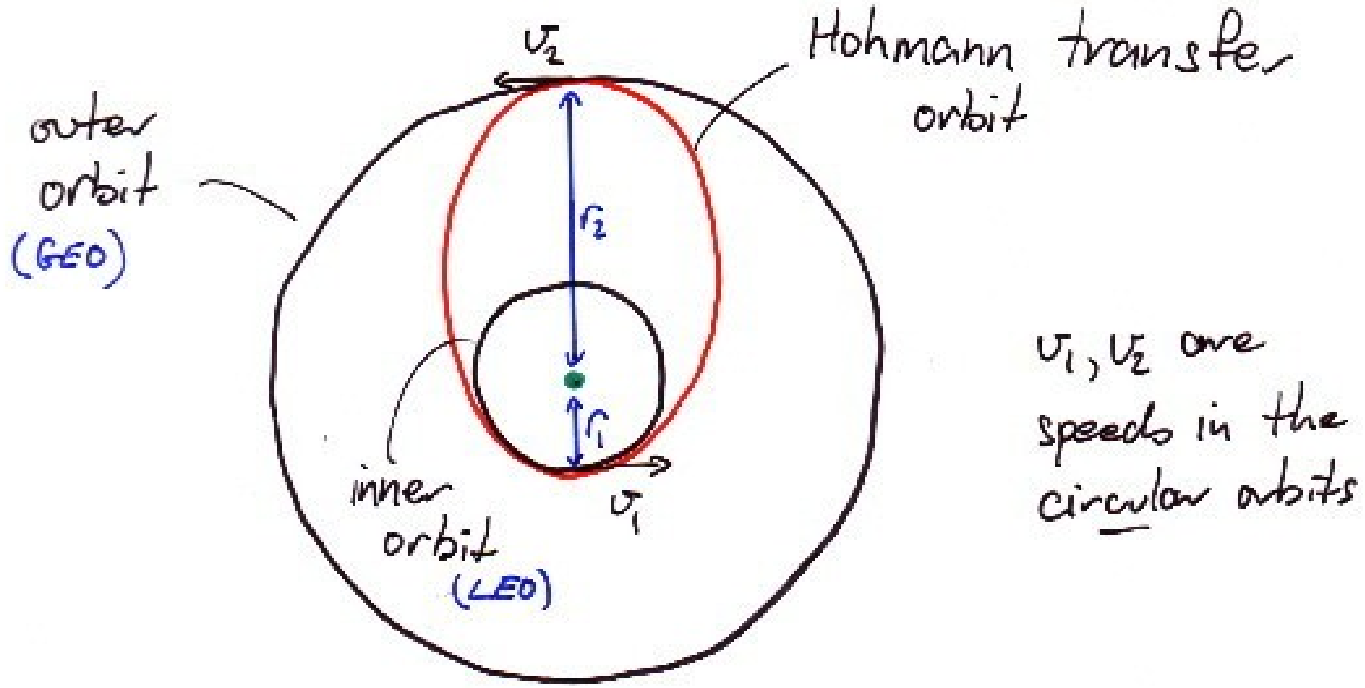
$= 611 \text{ km s}^{-1}$

Transfer orbits

Imagine we have a satellite in a circular low Earth orbit (or 'LEO'), and we want to transfer it into a higher orbit, such as a geosynchronous orbit (or 'GEO').

The most fuel efficient way to carry out the transfer is via a Hohmann transfer orbit.

This is an elliptical orbit which just touches the inner and outer circular orbits



Clearly, for the transfer orbit, its major axis is

$$2a_T = r_1 + r_2$$

We know that the energy of the transfer orbit is

$$E = -\frac{GMm}{2a_T}$$

M = mass of Earth
m = " " sat.

so larger $a_T \Rightarrow$ larger (more +ve) E

[remember, E is -ve in bound orbit]

The Hohmann transfer orbit has the least possible a, & $\therefore$ the least possible $E \Rightarrow$ most energy efficient.

- Note: only ½ the orbit is completed!

The transfer is done in 2 stages:

Stage #1

Boost the speed of the satellite in LEO, to inject it into the transfer orbit.

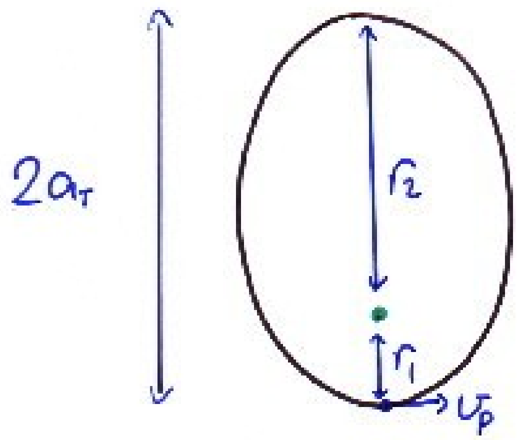
We will use the general equation

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right)$$

For the LEO, $r = a = r_1$

$$\Rightarrow \quad v_1^2 = \frac{GM}{r_1}$$

consider the transfer orbit :



apogee $= r_2 = a_T(1+e)$

perigee $= r_1 = a_T(1-e)$

$r_1 + r_2 = 2a_T$

velocity needed at perigee, v_p is

$$v_p^2 = GM\left(\frac{2}{r_1} - \frac{1}{a_T}\right)$$

$$= GM\left(\frac{2}{r_1} - \frac{2}{r_1+r_2}\right)$$

ie $$v_p^2 = 2GM\frac{r_2}{r_1(r_1+r_2)}$$

$$= v_1^2 \cdot \frac{2r_2}{r_1+r_2}$$

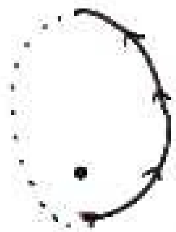
Since $r_2 > r_1 \Rightarrow v_p > v_1$

So we need to boost the speed of the satellite to inject it into the transfer orbit, by

$$\Delta v_1 = v_p - v_1 = \sqrt{\frac{GM}{r_1}}\left(\sqrt{\frac{2r_2}{r_1+r_2}} - 1\right)$$

This Δv ('delta-vee') is produced by a short burn on the rocket (note the boost is in the direction of motion) – see later...

- The satellite is now in the correct transfer orbit, with $a_T = \dfrac{r_1 + r_2}{2}$. It will reach apogee in half the orbital period:



$$\text{period, } T = \left(\frac{4\pi^2 a_T^3}{GM}\right)^{1/2} \qquad (K\text{III})$$

$$\Rightarrow \text{transfer time} = \frac{T}{2} = \left[\frac{\pi^2 (r_1 + r_2)^3}{8\,GM}\right]^{1/2}$$

Stage #2

Once the satellite reaches apogee, we need a further increase in speed, Δv_2, to inject it into its new circular orbit, radius r_2.

$$(\text{velocity needed})^2 = v_2^2 = \frac{GM}{r_2} \qquad [\text{as for } v_1]$$

At apogee of transfer orbit, velocity is

$$v_a^2 = GM\left(\frac{2}{r_2} - \frac{1}{a_T}\right)$$

$$\Rightarrow \quad v_a^2 = \frac{GM}{r_2}\,\frac{2r_1}{(r_1+r_2)} = v_2^2\,\frac{2r_1}{(r_1+r_2)}$$

(as $r_2 > r_1$, $v_a < v_2$, ie we do indeed need to boost the speed to perform the second orbit injection)

$$\Delta v_2 = v_2 - v_a = \sqrt{\frac{GM}{r_2}}\left(1 - \sqrt{\frac{2r_1}{r_1+r_2}}\right)$$

Following this burn, the satellite will perform circular orbits of radius r_2.

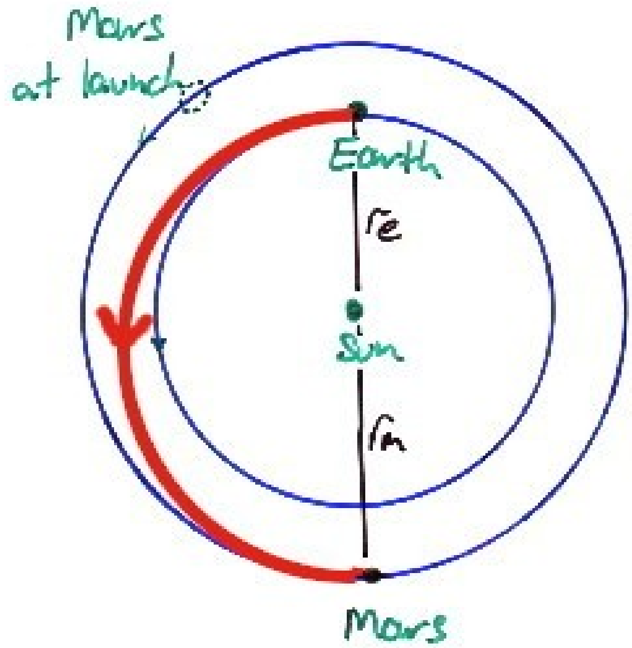
- The Hohmann transfer ellipse is also appropriate for interplanetary missions
- It minimises the fuel required (just two small burns) but not the trip time $\Rightarrow$ not appropriate for manned missions (?)

Example

The Mars Global Surveyor spacecraft was required to get from low Earth orbit to Mars. How was this done?:

- We will neglect the gravitational fields of the planets, and determine the Hohmann transfer orbit from a circular orbit about the Sun of radius 1AU (Earth) to 1.52AU (Mars)

$r_e = 1\,AU$
$r_m = 1.52\,AU$



As before, $a_T = \dfrac{r_e + r_m}{2} = 1.26\ AU$

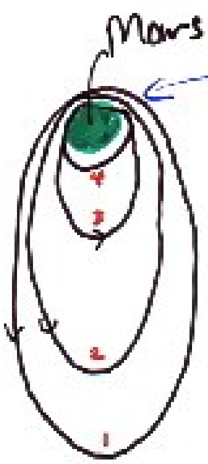
transfer time $= \frac{1}{2} a_T^{3/2} = 0.71\,yr = $ 8.5 months

↑ only ½ an orbit

- Mars must be in the correct location when the spacecraft arrive ⇒ idea of a launch window (in time).

- Actual travel time was 10 months. The transfer orbit used was slightly non-Hohmann because:
  - The orbit of mars is elliptical
  - We have neglected the gravitational effect of the planets (such as the 'gravity assist' of Mars - see later)

- The final burn must inject the spacecraft into orbit around mars. Aerobraking was used to make an initially elliptical orbit circular.



successive orbits

spacecraft is in atmosphere here.

⇒ friction
⇒ reduction of periapse velocity
⇒ reduction of eccentricity

After many passes, the orbit was circular.

Gravity Assist

Sometimes called 'the slingshot effect' or 'flyby'.

- Many interplanetary spacecraft (sc) missions use 'gravity assist' to accelerate the sc up to its highest speeds. Engines are only rarely used once in interplanetary space.

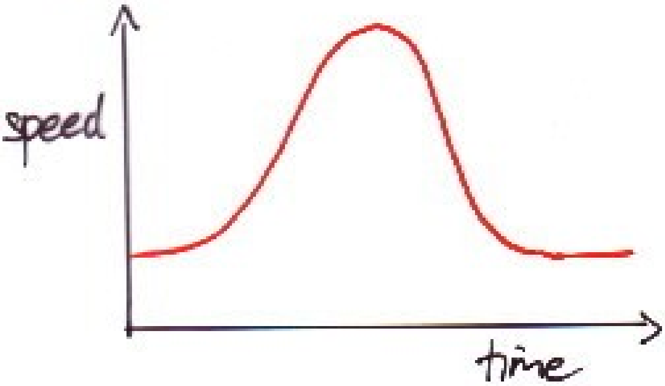
How it works:

Think of a spacecraft approaching a stationary planet at greater than the escape speed:

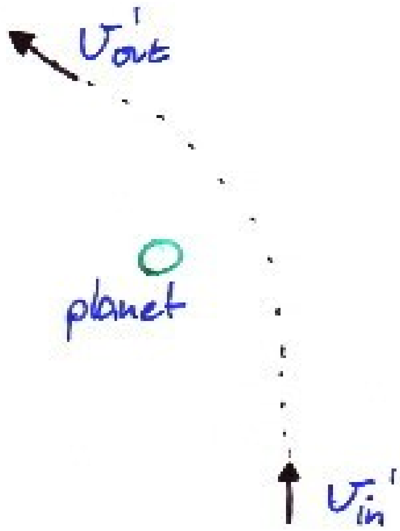
The trajectory of the sc is deflected into a hyperbola

$[E_{tot} > 0 \; ; \; e > 1]$

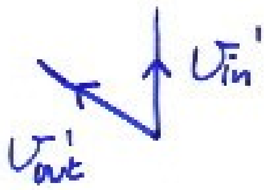
The sc accelerates as it is attracted to the planet, and decelerates as it flies away.



The magnitude of the velocity (≡ speed) returns to its initial value, but the direction of travel changes



as vectors:



[the prime (') means with respect to the planet]

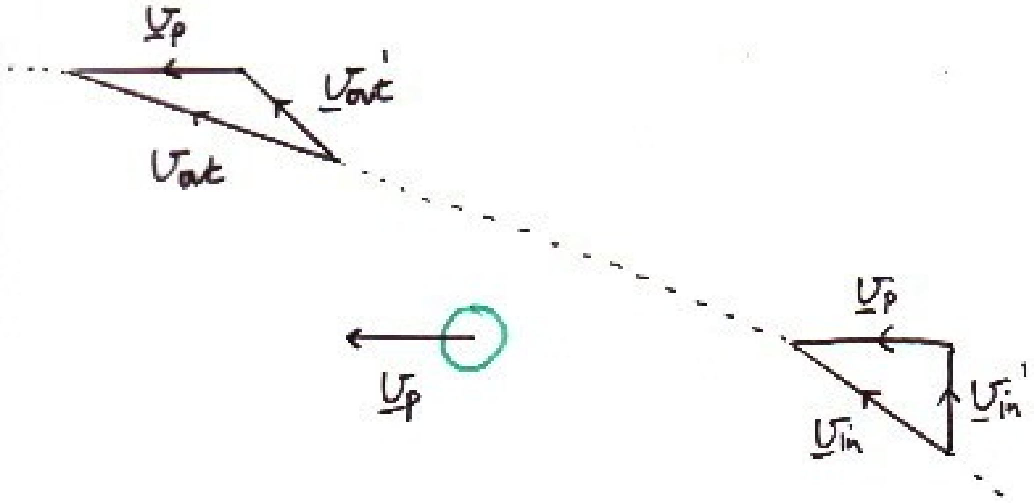
- This is a change in direction, but not a change in speed.
  - but what if the planet is moving ?

- If the planet is moving with a velocity $\underline{v}_p$, the velocity of the sc before ($\underline{v}_{in}$) and after ($\underline{v}_{out}$) the encounter is simply the vector sum:

$$\underline{v}_{in} = \underline{v}_{in}' + \underline{v}_p$$

$$\underline{v}_{out} = \underline{v}_{out}' + \underline{v}_p$$

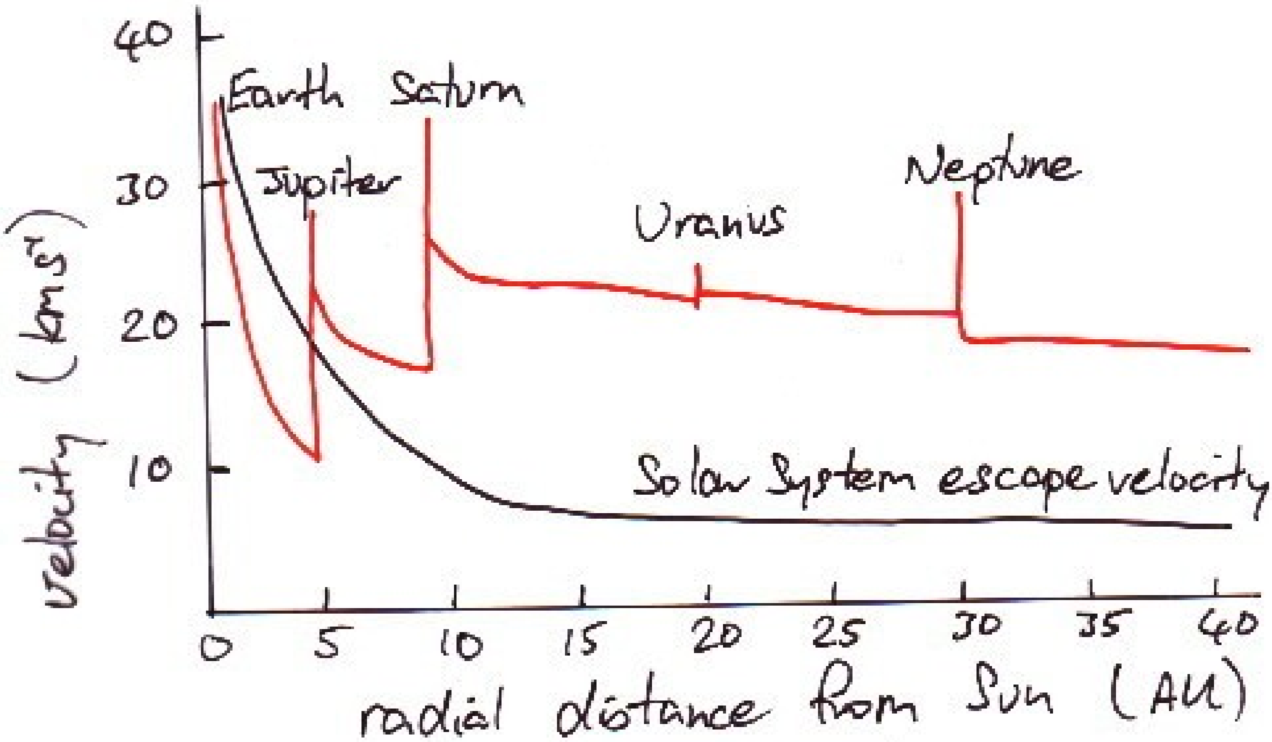
ie



- In this example, $\underline{v}_{out}$ is clearly greater than $\underline{v}_{in}$ – the sc has been speeded up by the encounter, as the planet has dragged it along, close to its direction of travel.
- The encounter will slightly slow the planet – momentum is transferred from the planet to the sc.

Examples

Voyager 2 carried out two major gravity assist flybys of Jupiter & Saturn, allowing it to reach solar escape velocity :



Cassini (designed to travel to Saturn) used a complicated Venus-Venus-Earth-Jupiter swingby sequence to boost it on to Saturn.

This saved 75 tons of rocket fuel, but increased the journey time to 7 years!

- Gravity assist can also be used to slow a sc, and insert it into orbit.

pros: • Need less fuel ⇒ cheaper

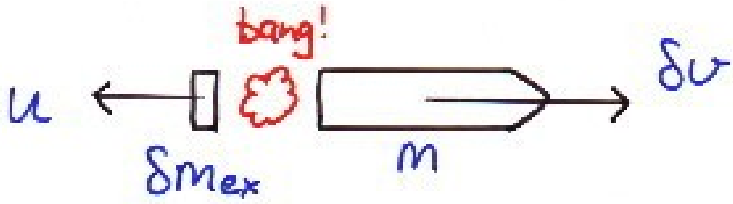
cons: • Often a very indirect route (∴ time-consuming)

• The planets need to be correctly placed (⇒ limited launch window)
The Voyager flyby sequence is only available every 175 years (we were lucky!)

The Rocket Equation

- How much fuel do you need to give a rocket an increase of speed δv ? [or indeed a decrease]

  - The problem is made harder because the rocket loses mass as fuel is burned.



- The rocket ejects a mass of exhaust δm_{ex} at a speed u relative to the rocket.
- The rocket + remaining fuel (total mass m) moves forward with a speed δv.

Conserving momentum :

$$u\,\delta m_{ex} = m\,\delta v$$

The rocket + fuel gets less massive by $\delta m = -\delta m_{ex}$

so $$-u\,\delta m = m\,\delta v$$

ie $$\delta v = -u\,\frac{\delta m}{m}$$

If the rocket + fuel starts at rest ($v=0$) with mass m_i, and ends (after burning the fuel) with a mass m_f, travelling at v, then

$$\int_0^v dv = -u \int_{m_i}^{m_f} \frac{dm}{m}$$

$$\Rightarrow \qquad v = u \ln \frac{m_i}{m_f}$$

This is the free space rocket equation

Example

Cassini's Jupiter flyby gave it a Δv of $2{\cdot}2\,\mathrm{km\,s^{-1}}$.

Given: mass of spacecraft = 5574 kg

engine exhaust velocity = $2\,\mathrm{km\,s^{-1}}$

How much fuel did this save?

Ans :

use $$\Delta v = u \ln\left[\frac{M_{cass} + M_{fuel}}{M_{cass}}\right]$$

ie $$2.2 = 2 \ln\left[1 + \frac{M_{fuel}}{5574}\right]$$

$$1 + \frac{M_{fuel}}{5574} = e^{1.1}$$

$$\Rightarrow \quad M_{fuel} = 11171 \text{ kg}$$ (~2x mass of space-craft!)

Even more fuel would be necessary to get this fuel to Jupiter distances in the first place.

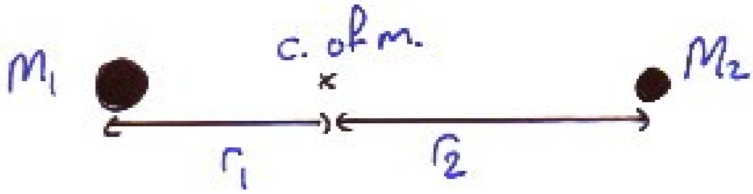
- Flybys are good things !

The Two-Body Problem

Until now, we have only considered the situation where the orbiting mass is much less than the central mass:

$m_1 \ll m_2$, so m_2 hardly moves.

- If $m_1 \sim m_2$ both masses move appreciably. However the centre of mass remains stationary:



$$m_1 r_1 = m_2 r_2$$

[This is the 'centre of mass frame'.

Total momentum = constant = 0.]

m_1 feels the normal gravitational force:

$$F = \frac{G m_1 m_2}{(r_1 + r_2)^2}$$

Using $\quad r_2 = \frac{m_1}{m_2} r_1$

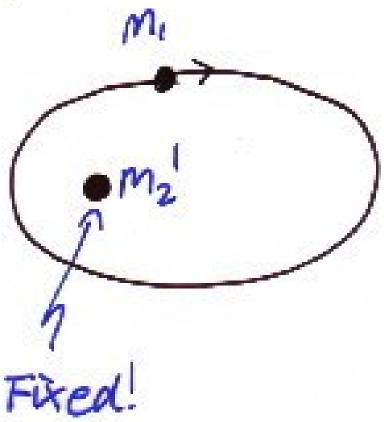
we can rewrite this as

$$F = \frac{G m_1 m_2}{\left(r_1 + r_1 \frac{m_1}{m_2}\right)^2} = \frac{G m_1}{r_1^2} \frac{m_2^3}{(m_1 + m_2)^2}$$

Now define $m_2' = \frac{m_2^3}{(m_1 + m_2)^2}$

$$\Rightarrow \qquad F = \frac{G m_1 m_2'}{r_1^2}$$

So m_1 will move as if attracted by a mass m_2', stationary at the centre of mass:



where $m_2' = \frac{m_2^3}{(m_1 + m_2)^2}$

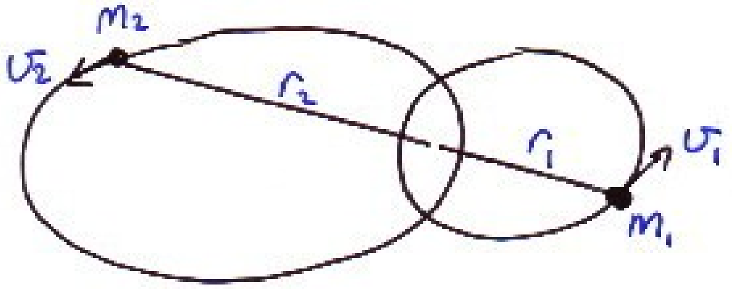
- This is an inverse-square law of attraction, so m_1 performs elliptical orbits about the centre of mass.

- Similarly, m_2 performs elliptical orbits about this point, as if attracted by a mass $m_1' = \dfrac{m_1^3}{(m_1+m_2)^2}$ there.

- At any instant the total momentum is zero, so

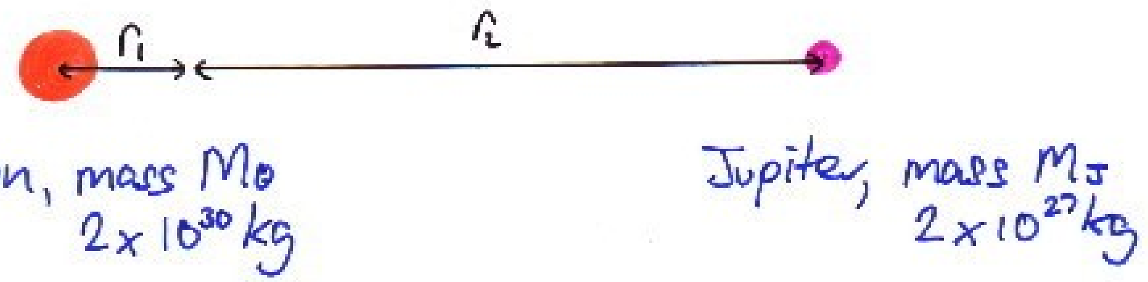
$$m_1 \underline{v}_1 + m_2 \underline{v}_2 = 0$$

The motion is something like:



m_1 + m_2 lie on a line passing through the centre of mass.

These corrections need to be considered even for planetary motion:



Jupiter orbits at a distance $\simeq 5.2\,AU = 7.8\times10^{11}$ m

$$M_\odot r_1 = M_J r_2$$

$$\Rightarrow \qquad r_1 = \frac{M_J}{M_\odot} r_2 \simeq 10^{-3} r_2 = 7.8\times10^{8}\,\text{m}$$

But the radius of the Sun is $\simeq 7\times10^{8}$m, so the centre of gravity is just above the Sun's surface.

- The Sun *is* moved around appreciably by Jupiter.

Relative motion.

It is sometimes convenient to think in terms of the relative separation of the two bodies,

$$r = r_1 + r_2$$

→ See handout...

Relative motion

The general elliptical analysis is *just* beyond the scope of this course (though not by much). Instead we will analyse the case of two stars of similar mass performing circular orbits about a common centre (Figure 1).

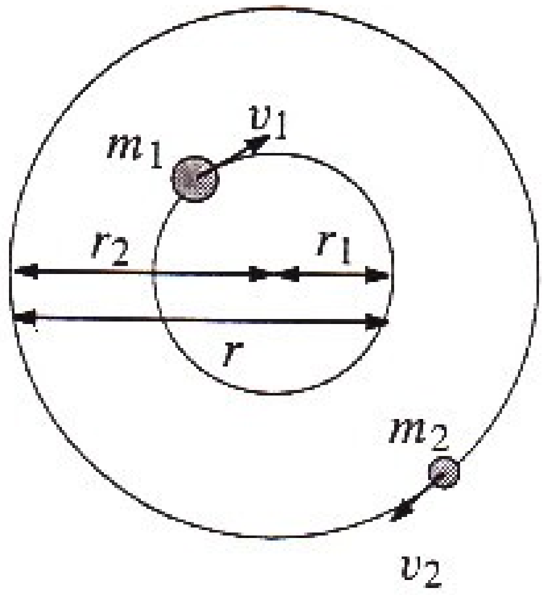


Figure 1: Masses m_1 and m_2 in circular orbits ($m_1 > m_2$).

Let the stars have masses m_1 and m_2, at distances r_1 and r_2 from their common centre. We will define the *radius vector* or the system as

$$r = r_1 + r_2.$$

r is therefore the relative separation of the stars which is constant in our example, but would vary if the orbits were elliptical. The two stars must have the same angular speed ω (otherwise one mass would catch up the other, and the gravitational force would not be directed towards the centre of the circles). The orbital speeds of the two masses are therefore

$$v_1 = \omega r_1,$$
$$v_2 = \omega r_2.$$

The gravitational force between the stars depends on the inverse-square of their separation, r, and supplies the centripetal force that keeps them moving in circles of radii r_1 and r_2 respectively. So

$$\frac{Gm_1m_2}{r^2} = \frac{m_1v_1^2}{r_1} = m_1\omega^2 r_1 \qquad (1)$$
$$= \frac{m_2v_2^2}{r_2} = m_2\omega^2 r_2. \qquad (2)$$

The equality of the two right-hand terms gives

$$\frac{r_1}{r_2} = \frac{m_2}{m_1},$$

which is just the condition that the common centre of the circles is the centre of mass of the system. Using Equations 1 and 2 we can also write

$$\omega^2 = \frac{Gm_1}{r^2 r_2} = \frac{Gm_2}{r^2 r_1}. \qquad (3)$$

Also, since

$$r = r_1 + r_2 = r_1 + \frac{m_1}{m_2} r_1,$$

we get

$$r_1 = \frac{m_2 r}{m_1 + m_2}.$$

Combining this with Equation 3 we get

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3}.$$

We can write this in terms of the period of the orbit, $T = 2\pi/\omega$, as

$$T^2 = \frac{4\pi^2 r^3}{G(m_1 + m_2)},$$

i.e.,

$$\boxed{\frac{r^3}{T^2} = \frac{G(m_1 + m_2)}{4\pi^2}.} \quad *$$

This is Kepler's third law for circular orbits. It is always true, even when one mass is much greater than the other, but is especially useful when $m_1 \simeq m_2$. If $m_1 \gg m_2$, then $r_2 \simeq r$ and the equation reduces to the 'one-body' version derived earlier in the course.

In fact all the earlier results can be applied to the two-body problem with the following trick (you will be shown the proof in honours astronomy!). We can define

$$
\begin{aligned}
M &= m_1 + m_2 \\
\mu &= \frac{m_1 m_2}{m_1 + m_2} \\
v &= v_1 + v_2.
\end{aligned}
$$

The quantity μ is called the *reduced mass* of the system. Note that if $m_1 \gg m_2$ then $M \simeq m_1$ and $\mu \simeq m_2$.

In general, the two-body problem can be treated as an equivalent one-body problem in which the reduced mass μ is orbiting about a fixed mass M at a distance r.

The reduced mass will describe an imagined ellipse about M with semi-major axis $a = a_1 + a_2$ and eccentricity e (which is the same as the eccentricities of the two real orbits). The radius vector, r, tells us the relative separation of the two masses. Here are some results (derived using the above rule) for the two-body system. Note their similarity to the one-body results derived earlier in the course, but remember the definitions of M, μ, v, a and r in terms of the parameters of the two stars:

Energy:	$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$
Angular momentum:	$L = \mu r v \sin\theta$
Speed:	$v^2 = GM(2/r - 1/a)$
Period:	$T^2 = \frac{4\pi^2 a^3}{GM}.$