Proof of Kepler's laws from Newtonian dynamics

It would be a pity to have a course on dynamical astronomy and not at least *see* a proof of Kepler's laws from Newton's laws of motion and gravitation. Importantly, **these proofs are not examinable!** They are presented here purely to satisfy curiosity and for your entertainment.

KEPLER'S FIRST LAW

KI: *A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse.* Take a mass m in a general

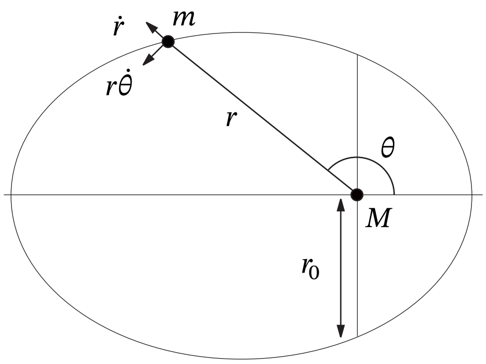

Figure 1: The geometry used in the proof.

elliptical orbit around a much more massive body M. When the separation of the masses is r the total energy of the orbit is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}, \tag{1}$$

where v is the speed of the orbiting mass and G the constant of gravitation. The velocity of m has two components: a radial component equal to $\mathrm{d}r/\mathrm{d}t$ (written $\dot{r}$) and a component perpendicular to r which is the 'circular' component of the velocity, equal to $r\omega$ where ω ($\equiv \dot{\theta}$) is the instantaneous angular velocity of the body, with θ as shown in Fig. 1. Because these components are orthogonal, the square of the total velocity equals the sum of the squares of these components. We can now write this energy equation in *polar* coordinates:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}. \tag{2}$$

Similarly, we can write the angular momentum of m as

$$L = mr^2\dot{\theta}, \tag{3}$$

as $r\dot{\theta}$ is the component of v perpendicular to r. Now make the substitution $\rho = 1/r$, so that $\dot{\theta} = L\rho^2/m$. We can evaluate θ as

$$\theta = \int \frac{L}{m}\rho^2\,\mathrm{d}t \tag{4}$$

$$= \int \frac{L}{m}\rho^2\frac{\mathrm{d}t}{\mathrm{d}\rho}\,\mathrm{d}\rho \tag{5}$$

$$\text{but} \quad \dot{r} = -\frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} \tag{6}$$

$$\text{so} \quad \theta = -\int \frac{L}{m\dot{r}}\,\mathrm{d}\rho. \tag{7}$$

Rearranging Eqn. (2) we can see that $\dot{r}$ is

$$\dot{r}^2 = \frac{2E}{m} + 2GM\rho - \frac{L^2}{m^2}\rho^2. \tag{8}$$

Now a further manipulation. We make the substitutions

$$r_0 = \frac{L^2}{GMm^2} \tag{9}$$

$$e^2 = 1 + \frac{2Er_0}{GMm}. \tag{10}$$

Clearly, both r_0 and e are constants. We choose them in this way so that our answer is immediately recognisable as an ellipse—it is not an obvious substitution at this stage! After a little manipulation, Eqn. (8) can be written as

$$\dot{r} = \frac{L}{m}\left[\frac{e^2}{r_0^2} - \left(\rho - \frac{1}{r_0}\right)^2\right]^{1/2}. \tag{11}$$

Substituting this into Eqn. (7) we get

$$\theta = -\int \frac{1}{\sqrt{(e/r_0)^2 - (\rho - 1/r_0)^2}}\,\mathrm{d}\rho \tag{12}$$

$$= \cos^{-1}\left(\frac{\rho - 1/r_0}{e/r_0}\right). \tag{13}$$

This can be rearranged to give

$$r = r_0/(1 + e\cos\theta), \tag{14}$$

which is the equation of an ellipse in polar coordinates, with the origin at a focus. We can now identify r_0 as the semi-latus rectum of the ellipse and e as its eccentricity.

KEPLER'S SECOND LAW

Let's now prove KII: *The line joining a planet to the Sun sweeps out equal areas in equal intervals of time*. The proof of KII highlights the generality of the 'sweeping out area' rule for motion under any central force.

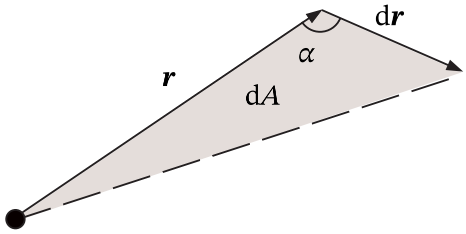

Figure 2: The area, dA, swept out in a time dt by $\boldsymbol{r}$.

In a time dt the planet will move by a small amount d$\boldsymbol{r}$. The small triangle this vector makes to the Sun (Fig. 2) has an area

$$\mathrm{d}A = \frac{1}{2} r \mathrm{d}r \sin\alpha, \tag{15}$$

where α is the angle between $\boldsymbol{r}$ and d$\boldsymbol{r}$ (remember the area of a triangle is ½ $ab \sin C$). This can be usefully written as a (pseudo)vector perpendicular to the plane of the triangle with magnitude dA using the vector cross-product:

$$\mathrm{d}\boldsymbol{A} = \frac{1}{2}\boldsymbol{r} \times \mathrm{d}\boldsymbol{r}. \tag{16}$$

The rate of sweeping out area due to movement is therefore

$$\dot{\boldsymbol{A}} = \frac{\mathrm{d}\boldsymbol{A}}{\mathrm{d}t} = \frac{1}{2}\boldsymbol{r} \times \dot{\boldsymbol{r}}. \tag{17}$$

Kepler's second law states that this is a constant for the orbital motion, so $\ddot{\boldsymbol{A}}$ should be zero if KII holds. Differentiating with respect to time again gives

$$\ddot{\boldsymbol{A}} = \frac{1}{2}\left(\dot{\boldsymbol{r}} \times \dot{\boldsymbol{r}} + \boldsymbol{r} \times \ddot{\boldsymbol{r}}\right). \tag{18}$$

The first term on the right-hand side certainly equals zero, as it is the cross-product of a vector with itself, but the second term is not zero for general motion. However, $\ddot{\boldsymbol{r}}$ is just the acceleration of the planet, and by Newton's second law that is in the direction of the applied (gravitational) force, so is also directed along $\boldsymbol{r}$. Therefore the second term must also be zero in this case. We can therefore say that $\ddot{\boldsymbol{A}} = 0$ and so $\dot{\boldsymbol{A}}$ is a constant. It's clear this would be true for any 'central force', where the force is directed along the line connecting the centres of mass.

KEPLER'S THIRD LAW

Now for KIII: *The square of the orbital period of a planet is proportional to the cube of the semi-major axis of its orbit.* We'll use some of the results from the lectures for this one.

The total area of an ellipse of semimajor and semiminor axes a and b is

$$A_{\mathrm{tot}} = \pi ab = \pi a^2\sqrt{1-e^2}, \tag{19}$$

where e is again the eccentricity (we derived $b = a\sqrt{1-e^2}$ in the lectures using the 'string' definition of an ellipse). From Eqn. (17), the rate of sweeping out area is

$$\dot{\boldsymbol{A}} = \frac{1}{2}\boldsymbol{r} \times \dot{\boldsymbol{r}} = \frac{1}{2}\boldsymbol{r} \times \boldsymbol{v} = \frac{\boldsymbol{L}}{2m}, \tag{20}$$

where $\boldsymbol{L}$ is the planet's orbital angular momentum around the Sun and m is its mass. The orbital period T is simply the time taken to sweep out an area A_{tot}, i.e.,

$$T = \frac{\pi ab}{L/(2m)} = \frac{m}{L} 2\pi a^2\sqrt{1-e^2}, \tag{21}$$

so

$$T^2 = \frac{m^2}{L^2} 4\pi^2 a^4 (1-e^2). \tag{22}$$

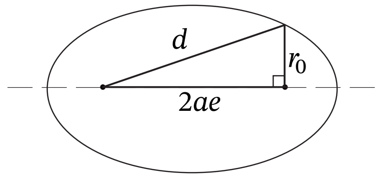

Figure 3: Relating r_0 to a and e.

We are nearly there, but we need to address the L and e terms in this expression. Using Fig. 3, and that $d + r_0 = 2a$ for an ellipse, we have (by Pythagoras)

$$(2a - r_0)^2 = 4a^2e^2 + r_0^2 \tag{23}$$

$$r_0 = a(1-e^2). \tag{24}$$

Inserting r_0 from Eqn. (9) we get

$$\frac{m^2}{L^2} = \frac{1}{GMa(1-e^2)}. \tag{25}$$

We can therefore write Eqn. (22) as

$$T^2 = \frac{4\pi^2}{GM} a^3, \tag{26}$$

which is KIII. As a bonus we get the constant of proportionality.

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