# Proof of Kepler's first law from Newtonian dynamics

## *A planet orbits the Sun in an ellipse, with the Sun at one focus of the ellipse.*

It would be a pity to have a course on dynamical astronomy and not at least *see* a proof of Kelper's first law from Newton's laws of motion and gravitation. Again, **this proof is not examinable!** It is presented here purely to satisfy curiosity and for your entertainment.

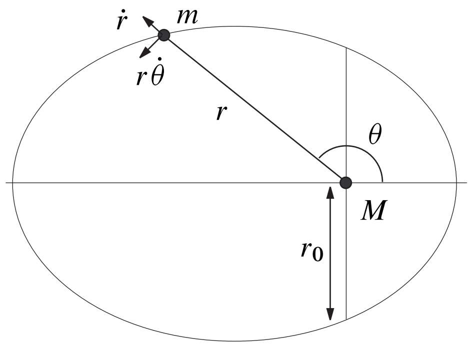


Figure 1: The geometry used in the proof.

Take a mass $m$ in a general elliptical orbit around a much more massive body $M$. When the separation of the masses is $r$, the total energy of the orbit is

$$E = \frac{1}{2}mv^2 - \frac{GMm}{r}, \tag{1}$$

where $v$ is the speed of the orbiting mass and $G$ the constant of gravitation. The velocity of $m$ has two components: a radial component equal to $\mathrm{d}r/\mathrm{d}t$ (written $\dot{r}$) and a component perpendicular to $r$ which is the 'circular' component of the velocity, equal to $r\omega$ where $\omega$ $(= \dot{\theta})$ is the instantaneous angular velocity of the body, with $\theta$ as shown in the diagram. Because these components are orthogonal, the square of the total velocity equals the sum of the squares of these components. We can now write this energy equation in *polar* coordinates:

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{GMm}{r}. \tag{2}$$

Similarly, we can write the angular momentum of $m$ as

$$L = mr^2\dot{\theta}, \tag{3}$$

as $r\dot{\theta}$ is the component of $v$ perpendicular to $r$. Now make the substitution $\rho = 1/r$, so that $\dot{\theta} = L\rho^2/m$. We can evaluate $\theta$ as

$$\theta = \int \frac{L}{m}\rho^2 \,\mathrm{d}t \tag{4}$$

$$= \int \frac{L}{m}\rho^2 \frac{\mathrm{d}t}{\mathrm{d}\rho}\,\mathrm{d}\rho \tag{5}$$

$$\text{but} \quad \dot{r} = -\frac{1}{\rho^2}\frac{\mathrm{d}\rho}{\mathrm{d}t} \tag{6}$$

$$\text{so} \quad \theta = -\int \frac{L}{m\dot{r}}\,\mathrm{d}\rho. \tag{7}$$

Rearranging equation 2 we can see that $\dot{r}$ is

$$\dot{r}^2 = \frac{2E}{m} + 2GM\rho - \frac{L^2}{m^2}\rho^2. \tag{8}$$

Now a further manipulation. We make the substitutions

$$r_0 = \frac{L^2}{GMm^2} \tag{9}$$

$$e^2 = 1 + \frac{2Er_0}{GMm}. \tag{10}$$

Clearly, both $r_0$ and $e$ are constants. We choose them in this way so that our answer is immediately recognisable as an ellipse – it is not an obvious substitution at this stage! After a little manipulation, equation 8 can be written as

$$\dot{r} = \frac{L}{m}\left[\frac{e^2}{r_0^2} - \left(\rho - \frac{1}{r_0}\right)^2\right]^{1/2}. \tag{11}$$

Substituting this into equation 7 we get

$$\theta = -\int \frac{1}{\sqrt{(e/r_0)^2 - (\rho - 1/r_0)^2}}\,\mathrm{d}\rho \tag{12}$$

$$= \cos^{-1}\left(\frac{\rho - 1/r_0}{e/r_0}\right). \tag{13}$$

This can be rearranged to give

$$r = r_0/(1 + e\cos\theta), \tag{14}$$

which is the equation of an ellipse in polar coordinates, with the origin at a focus. We can now identify $r_0$ as the semi-latus rectum of the ellipse and $e$ as its eccentricity.

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