Astronomy 345 Circumstellar Matter II
Problems 2012 (with solutions)

Answers to some numerical problems are shown in curly brackets. Solutions are shown immediately after their questions (which are in smaller type), with a horizontal line separating each problem.

speed of light	c	2.998×10^{8} m s^{-1}
gravitational constant	G	6.673×10^{-11} N m^2 kg^{-2}
Planck constant	h	6.626×10^{-34} J s
Boltzmann constant	k	1.381×10^{-23} J K^{-1}
Stefan-Boltzmann constant	σ	5.671×10^{-8} W m^{-2} K^{-4}
gas constant	R	8.315 J mol^{-1} K^{-1}
proton mass	m_{p}	1.673×10^{-27} kg
electron mass	m_{e}	9.109×10^{-31} kg
astronomical unit	AU	1.496×10^{11} m
Earth radius	$R_\oplus$	6.371×10^{6} m
solar mass	$M_\odot$	1.989×10^{30} kg
solar radius	$R_\odot$	6.960×10^{8} m
solar luminosity	$L_\odot$	3.826×10^{26} W
Thomson cross section	σ_{T}	6.652×10^{-29} m^2
plasma conduction constant	κ_0	1.2×10^{-12} W m^{-1} K$^{-7/2}$

1) Show that the atmosphere of a star in isothermal hydrostatic equilibrium has a number density profile of the form

$$n(r) = n_0 \exp\left(-a\left[1 - \frac{r_0}{r}\right]\right) ,$$

where n_0 is the number density at a radius r_0, and a is a constant. Derive an expression for a in terms of the star's atmospheric temperature and mass.

Solution to (1): Use

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -\frac{n m_{\mathrm{p}} G M_\odot}{r^2} \quad \text{and} \quad p = 2nkT.$$

Noting that T is constant, write as

$$\frac{\mathrm{d}n}{\mathrm{d}r} = -\frac{G M_\odot m_{\mathrm{p}}}{2kT}\frac{n}{r^2} = -\alpha\frac{n}{r^2}.$$

Integrating:

$$\int_{n_0}^{n} \frac{\mathrm{d}n}{n} = -\alpha \int_{r_0}^{r} \frac{\mathrm{d}r}{r^2}$$

$$\ln \frac{n}{n_0} = -\alpha \left(\frac{1}{r_0} - \frac{1}{r} \right)$$

$$n = n_0 \exp \left[-\frac{\alpha}{r_0} \left(1 - \frac{r_0}{r} \right) \right]$$

$$\text{i.e.,} \quad a = \frac{\alpha}{r_0} = \frac{GM_\odot m_\mathrm{p}}{2kTr_0}.$$

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2) The lower solar corona can be modelled as being in isothermal hydrostatic equilibrium at a temperature T, although the model breaks down at large distances from the Sun.

(a) Show that for radial distances $r = R_\odot + z$, where $R_\odot$ is the radius of the Sun, the number density of protons is

$$n(r) \simeq n_0 \exp(-z/h) , \quad \text{where} \quad h = \frac{2kTR_\odot^2}{GM_\odot m_\mathrm{p}} , \quad \text{and} \quad z \ll R_\odot ,$$

just as in a plane stratified isothermal atmosphere.

(b) Using this approximation for $n(z)$, show that the optical depth of the corona due to Thomson scattering is $\tau_\mathrm{s} \simeq n_0 h \sigma_\mathrm{T}$, where σ_T is the Thomson scattering cross section. Calculate τ_s assuming $T = 1.5 \times 10^6$ K and $n_0 = 10^{15}\ \mathrm{m}^{-3}$.*

(c) Combine the results above to show that the *volume emission measure*, $\int_\mathrm{vol} n^2\ \mathrm{d}V$, of the corona above $r = R_\odot$ is approximately $2\pi n_0^2 R_\odot^2 h$. (You may assume that $\int_a^\infty x^b e^{-x}\ \mathrm{d}x \simeq a^b e^{-a}$ for $a \gg 1$.)

Solution to (2):

(a) From the previous question, $n(r) = n_0 \exp\left[-a(1 - r_0/r)\right]$. If $r = R_\odot + z$, then $1 - r_0/r = 1 - R_\odot/(R_\odot + z) \simeq z/R_\odot$. i.e., $n(r) \simeq n_0 \exp(-az/R_\odot) = n_0 \exp(-z/h)$ where

$$h = \frac{R_\odot}{a} = \frac{2kTR_\odot^2}{GM_\odot m_\mathrm{p}} = 9.03 \times 10^7\ \mathrm{m}.$$

(b) $\tau = \int_{R_\odot}^\infty n(r)\sigma_\mathrm{T}\ \mathrm{d}r = \int_0^\infty n_0 \sigma_\mathrm{T} \mathrm{e}^{-z/h}\ \mathrm{d}z = n_0 \sigma_\mathrm{T} h = 6.01 \times 10^{-6}$

(c) From the definition of volume emission measure (VEM) and from the above, define

*Because we see the corona in scattered light, this value of τ_s is roughly the ratio of the surface brightness of the corona as seen during an eclipse to the normal surface brightness of the photosphere.

$x = r/R_\odot$ and $y = 2R_\odot x/h$:

$$\begin{aligned}
\mathrm{VEM} &= 4\pi \int_{R_\odot}^{\infty} n_0^2 \exp\left[-2(r - R_\odot)/h\right] r^2 \,\mathrm{d}r \\
&= 4\pi n_0^2 R_\odot^3 \int_1^{\infty} \exp\left[-2R_\odot(x-1)/h\right] x^2 \,\mathrm{d}x \\
&= 4\pi n_0^2 R_\odot^3 \mathrm{e}^{2R_\odot/h} \int_1^{\infty} \mathrm{e}^{-2R_\odot x/h} x^2 \,\mathrm{d}x \\
&= 4\pi n_0^2 R_\odot^3 \mathrm{e}^{2R_\odot/h} \int_{2R_\odot/h}^{\infty} \mathrm{e}^{-y} \frac{h^3}{8R_\odot^3} y^2 \,\mathrm{d}y \\
&\simeq \frac{\pi n_0^2}{2} h^3 \left(\frac{2R_\odot}{h}\right)^2 \\
&= 2\pi n_o^2 h R_\odot^2
\end{aligned}$$

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3) The Chapman model for a hydrostatic corona assumes that thermal conduction is the dominant heat loss mechanism from the plasma, giving it a temperature profile $T(r) \propto r^{-2/7}$. Use the equation of hydrostatic equilibrium to show that the number density profile in this atmosphere, $n(r)$, has the form

$$\frac{n(r)}{n_0} = \left(\frac{r}{r_0}\right)^{2/7} \exp\left\{\left(\frac{7}{10}\frac{GM_* m_\mathrm{p}}{kT_0 r_0}\right)\left[\left(\frac{r_0}{r}\right)^{5/7} - 1\right]\right\},$$

where the subscript '0' denotes the value of a parameter at a distance r_0 from the centre of the star.

Solution to (3):

$$\frac{\mathrm{d}p}{\mathrm{d}r} = -n m_\mathrm{p} \frac{GM_*}{r^2}; \quad p = 2nkT$$

$$\text{so} \quad 2k\frac{\mathrm{d}}{\mathrm{d}r}(nT) = -\frac{GM_* m_\mathrm{p} n}{r^2}.$$

$$\text{If} \quad T(r) \propto r^{-2/7}, \quad \frac{T}{T_0} = \left(\frac{r_0}{r}\right)^{2/7}$$

$$\text{so} \quad \frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n}{r^{2/7}}\right) = -\frac{GM_* n m_\mathrm{p}}{2kT_0 r_0^{2/7}} \frac{1}{r^2},$$

$$\text{i.e.,} \quad \frac{r^{2/7}}{n}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{n}{r^{2/7}}\right) = -\frac{GM_* m_\mathrm{p}}{2kT_0 r_0^{2/7}} r^{-12/7}$$

$$\ln\left(\frac{n}{r^{2/7}}\right) = \frac{7}{10}\frac{GM_* m_\mathrm{p}}{kT_0 r_0^{2/7}} r^{-5/7} + \text{const.}$$

$$\text{i.e.,} \quad \frac{n(r)}{n_0} = \left(\frac{r}{r_0}\right)^{2/7} \exp\left\{\left(\frac{7}{10}\frac{GM_* m_\mathrm{p}}{kT_0 r_0}\right)\left[\left(\frac{r_0}{r}\right)^{5/7} - 1\right]\right\}$$

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4) Show that for the Sun, the hydrostatic expression for $n(r)$ derived above for the Chapman model, has a maximum at

$$r_\mathrm{max} = R_\odot \left(\frac{7GM_\odot m_\mathrm{p}}{4kT_0 R_\odot}\right)^{7/5},$$

and calculate this value of $r_{\max}$ assuming the corona has a base temperature of $T_0 = 1.5 \times 10^6$ K.

Solution to (4): Putting $x = r/R_\odot$ and $\alpha = GM_\odot m_\mathrm{p}/(kTR_\odot)$, $n(r)$ has the form

$$\begin{aligned} n(x) &= n_0 x^{2/7} \exp\left[\frac{7}{10}\alpha(x^{-5/7}-1)\right], \\ \text{so} \quad \frac{\mathrm{d}n}{\mathrm{d}x} &= \frac{2}{7}n_0 x^{-5/7}\mathrm{e}^{[\cdot]} - n_0 x^{2/7}\frac{7\alpha}{10}\frac{5}{7}x^{-12/7}\mathrm{e}^{[\cdot]} \\ &= \frac{n_0}{2}x^{-10/7}\mathrm{e}^{[\cdot]}\left(\frac{4}{7}x^{5/7}-\alpha\right). \end{aligned}$$

$$\text{When} \quad \frac{\mathrm{d}n}{\mathrm{d}x} = 0, \quad x_{\max} = \frac{r_{\max}}{R_\odot} = \left(\frac{7}{4}\alpha\right)^{7/5} = \left(\frac{7GM_\odot m_\mathrm{p}}{4kTR_\odot}\right)^{7/5}.$$

$$\text{If } T_0 = 1.5 \times 10^6\,\text{K then } r_{\max} = 101R_\odot = 7.01 \times 10^{10}\,\text{m}.$$

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5) Use the Parker model for the velocity profile of an isothermal, pressure driven wind, together with the equation of mass conservation, to show that the number density profile in the wind, $n(r)$ satisfies

$$\left[\frac{n_\mathrm{c}}{n(r)}\right]^2\left(\frac{r_\mathrm{c}}{r}\right)^4 - \ln\left[\frac{n_\mathrm{c}}{n(r)}\right]^2 = 4\frac{r_\mathrm{c}}{r} - 3\,.$$

Solution to (5):

$$\begin{aligned} \text{Given} \quad & \frac{v^2}{v_\mathrm{c}^2} - 2\ln\frac{v}{v_\mathrm{c}} = 4\frac{r_\mathrm{c}}{r} + 4\ln\frac{r}{r_\mathrm{c}} - 3, \\ & \text{and mass continuity:} \quad nvr^2 = \text{const.}, \\ \text{we get} \quad & \left(\frac{n_\mathrm{c}}{n}\right)^2\left(\frac{r_\mathrm{c}}{r}\right)^4 - 2\ln\frac{n_\mathrm{c}}{n} = 4\frac{r_\mathrm{c}}{r} - 3 \\ \text{i.e.,} \quad & \left[\frac{n_\mathrm{c}}{n(r)}\right]^2\left(\frac{r_\mathrm{c}}{r}\right)^4 - \ln\left[\frac{n_\mathrm{c}}{n(r)}\right]^2 = 4\frac{r_\mathrm{c}}{r} - 3 \end{aligned}$$

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6) Determine the velocity gradient, $\mathrm{d}v/\mathrm{d}r$, in the solar wind at the critical (sonic) point of the Parker model, assuming that the wind velocity follows the critical solution, i.e., that the velocity equals the sound speed at the critical point, and l'Hôpital's rule: If $f(a) = g(a) = 0$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}\,.$$

Solution to (6): The parker model gives

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{\frac{2c^2}{r}\left(1-\frac{r_\mathrm{c}}{r}\right)}{v^2-c^2}$$

$$\text{i.e.,}\quad \frac{r^2}{2c^2 v}\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{r-r_\mathrm{c}}{v^2-c^2} = \frac{f}{g}.$$

$$f' = 1,\quad g' = 2v\frac{\mathrm{d}v}{\mathrm{d}r}, \text{ so at } v=c,\quad \frac{f}{g} = \frac{f'}{g'} = \frac{1}{2cv'}.$$

$$\text{So at } r_\mathrm{c},\quad \frac{r_\mathrm{c}^2 v'}{2c^3} = \frac{1}{2cv'},$$

$$v' = \pm\frac{c}{r_\mathrm{c}} = \pm\frac{2c^3}{GM_*}$$

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7) The masses, M, luminosities, L, and radii, R, of massive main sequence stars are related to their solar values by by

$$\frac{R}{R_\odot} = \frac{M}{M_\odot}; \qquad \frac{L}{L_\odot} = \left(\frac{M}{M_\odot}\right)^4.$$

Show that radiatively driven winds, comprising particles of mean (photon) cross-section σ and mass m, will occur in stars of mass $M \geq M_\mathrm{min}$, where

$$\frac{M_\mathrm{min}}{M_\odot} = \left(\frac{4\pi G M_\odot mc}{\sigma L_\odot}\right)^{1/3}.$$

Calculate $M_\mathrm{min}/M_\odot$ when $\sigma = \sigma_\mathrm{T}$ and $m = m_\mathrm{p}$.

Show also that the terminal wind speed from these massive stars is

$$v_\infty = v_0\left(\frac{M}{M_\odot}\right)^{3/2}\left[1-\left(\frac{M_\mathrm{min}}{M}\right)^3\right]^{1/2}, \quad \text{where} \quad v_0 = \left(\frac{\sigma L_\odot}{2\pi mcR_\odot}\right)^{1/2}.$$

Again, calculate v_∞ when $\sigma = \sigma_\mathrm{T}$ and $m = m_\mathrm{p}$.

Solution to (7): Critical luminosity, $L_\mathrm{c} = 4\pi GMmc/\sigma$, where M is the particle mass and σ is its radiation cross-section. Radiatively driven winds require $L \geq L_\mathrm{c}$, i.e.,

$$\frac{L}{L_\odot} \geq \frac{4\pi Gmc}{\sigma}\frac{M}{M_\odot}\frac{M_\odot}{L_\odot}.$$

But $L/L_\odot = (M/M_\odot)^4$ so

$$\left(\frac{M}{M_\odot}\right)^4 \geq 4\pi G\frac{M}{M_\odot}\frac{M_\odot}{L_\odot}\frac{mc}{\sigma}$$

$$\text{i.e.,}\quad M \geq M_\mathrm{min} = M_\odot\left(\frac{4\pi GM_\odot mc}{\sigma L_\odot}\right)^{1/3}$$

Putting $\sigma = \sigma_T$ and $m = m_p$, $M_{min}/M_\odot = 31.7$.

$$
\begin{aligned}
\text{If} \quad v_\infty &= \left[\frac{(L - L_c)\sigma}{2\pi Rmc}\right]^{1/2}; \; v_0^2 = \frac{\sigma L_\odot}{2\pi mcR_\odot} \\
\text{then} v_\infty &= v_0 \left[\left(\frac{L}{L_\odot} - \frac{L_c}{L_\odot}\right)\frac{R_\odot}{R}\right]^{1/2} \\
&= v_0 \left[\left\{\left(\frac{M}{M_\odot}\right)^4 - \frac{4\pi GMmc}{\sigma L_\odot}\right\}\frac{M_\odot}{M}\right]^{1/2} \\
&= v_0 \left(\frac{M}{M_\odot}\right)^{3/2}\left[1 - \frac{4\pi GM_\odot mc}{\sigma L_\odot}\left(\frac{M_\odot}{M}\right)^3\right]^{1/2} \\
&= v_0 \left(\frac{M}{M_\odot}\right)^{3/2}\left[1 - \left(\frac{M_{min}}{M}\right)^3\right]^{1/2} = 3.46\,\mathrm{km\,s^{-1}}
\end{aligned}
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8) Given the mean solar wind speed at 1 AU is about 350 km s^{-1}, and its mean proton number density is about 3 cm^{-3}, compare the dynamic (ram) pressure of the solar wind at this distance with the photon pressure. Estimate the force each exerts on the Earth.

Solution to (8): At 1 AU, the solar wind momentum density is nm_pv, so the momentum flux (i.e., pressure) $= nm_pv^2 \simeq 6 \times 10^{-10}$ Pa.

For the photon pressure, assume all photons are emitted at an angular frequency ω. The solar luminosity, $L_\odot = n\hbar\omega$, where n is the number of photons emitted by the Sun per unit time. The photon flux at R is therefore

$$F_n = \frac{n}{4\pi R^2} = \frac{L_\odot}{\hbar\omega}\frac{1}{4\pi R^2}.$$

The momentum of each photon $= \hbar k = \hbar\omega/c$, so the photon pressure is

$$
\begin{aligned}
P &= \frac{\hbar\omega}{c}\frac{L_\odot}{\hbar\omega}\frac{1}{4\pi R^2} \\
&= \frac{L_\odot}{c}\frac{1}{4\pi R^2} \simeq 4.5 \times 10^{-6}\,\mathrm{Pa}.
\end{aligned}
$$

The ratio of photon:wind ram pressures $= nm_pv^2c4\pi R^2/L_\odot \simeq 7\,500$.

The cross-section of the Earth $= \pi R_\oplus^2 \simeq 1.3 \times 10^{14}\,\mathrm{m}^2$, so photon force $\sim 6 \times 10^8$ N, and solar wind force $\sim 8 \times 10^5$ N.

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9) A star has a continuous Planck spectrum at frequencies up to the Lyman continuum limit of hydrogen, and is very weak at higher frequencies due to opacity effects. Hydrogen atoms in the stellar atmosphere are accelerated radiatively by absorbing stellar continuum radiation in their Lyman-α line. By considering the Doppler shift of the radiation absorbed show that, no matter how luminous the star, the terminal speed of the stellar wind cannot exceed $c/4$. (Neglect relativistic effects.)

Solution to (9): Use the Rydberg equation,

$$\frac{1}{\lambda} = R\left(\frac{1}{n_1^2} - \frac{1}{n_2^2}\right) = \frac{\nu_n}{c}.$$

For the Lyman series, $n_1 = 1$.

$$\text{Lyman limit: } \nu_\infty = Rc(1-0) = Rc$$
$$\text{Lyman alpha: } \nu_\alpha = Rc(1-1/4) = 3Rc/4.$$

Combining the above, $\nu_\alpha/\nu_\infty = 3/4$. As atoms accelerate away from the star, the emission spectrum, F_ν, appears redshifted. Once the highest frequency, ν_∞, is shifted below the absorption frequency, ν_α, there can be no further absorption and therefore no further acceleration of the wind. This occurs at a wind speed of

$$\begin{aligned} v &= c\frac{\Delta\nu}{\nu} = \frac{c(\nu_\infty - \nu_\alpha)}{\nu_\infty} \\ &= c\left(1 - \frac{\nu_\alpha}{\nu_\infty}\right) \\ &= \frac{c}{4}. \end{aligned}$$

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10) (Exam'98) What are the major assumptions behind the Parker model for the solar wind? [4]

The critical solution for this model has the form

$$\frac{v^2}{c^2} - 2\ln\frac{v}{c} = 4\ln\frac{r}{r_c} + 4\frac{r_c}{r} - 3$$

where v is the wind speed at a radial distance r, $c = (2kT/m_p)^{1/2}$ is the isothermal sound speed and $r_c = GM_\odot/(2c^2)$ is the critical point. Show that if $r \gg r_c$, then the wind speed is approximately

$$v \simeq 2c\sqrt{\ln(r/r_c)},$$

increasing by only about 30 percent between 20 and 1000 AU. (You may assume the temperature of the wind is 6×10^5 K.) [6]

Assuming a constant mass loss rate from the Sun, show that the dynamical pressure of the wind (i.e., the wind momentum flowing through unit area in unit time) at large ($\sim$ 100 AU) distances from the Sun is

$$P(r) = \frac{r_0^2 n_0 v_0 m_p v(r)}{r^2} \simeq \frac{10^{-9}}{(r/r_0)^2} \quad \text{pascal},$$

where r_0 and v_0 are values taken at 1 AU, and the proton number density at 1 AU, $n_0 = 3 \times 10^6\ \mathrm{m^{-3}}$. [5]

It is assumed that the solar wind extends to the point where this dynamical pressure equals the hydrostatic pressure of the interstellar medium. Given the proton number density in the interstellar medium is about $5 \times 10^5\ \mathrm{m^{-3}}$ and that its temperature is about 8 000 K, estimate the extent of the solar wind in AU. [5]

Solution to (10): Assumptions behind the Parker wind model: isotropic isothermal corona/wind. Wind driven by thermal pressure alone (no photon pressure). Using the given equation for the velocity profile, set $r \gg r_c$ and $v \gg c$ (i.e., supersonic). The equation reduces to $v^2/c^2 \simeq 4\ln(r/r_c)$, so $v \simeq 2c\,[\ln(r/r_c)]^{1/2}$.

$$c = (2kT/m_p)^{1/2} = (2 \times 1.38 \times 10^{-23} \times 6 \times 10^5/1.67 \times 10^{-27})^{1/2}$$
$$= 9.96 \times 10^4 \simeq 10^5\,\mathrm{m\,s^{-1}}.$$
$$r_c = GM_\odot/(2c^2) = 6.67 \times 10^{-11} \times 2 \times 10^{30}/2 \times 10^{10} = 6.67 \times 10^9\,\mathrm{m}$$
$$\text{hence}\quad v_{20\,\mathrm{AU}} = 2 \times 10^5 \left[\ln\left(\frac{20 \times 1.5 \times 10^{11}}{6.67 \times 10^9}\right)\right]^{1/2} = 494\,\mathrm{km\,s^{-1}}$$
$$v_{1000\,\mathrm{AU}} = 2 \times 10^5 \left[\ln\left(\frac{10^3 \times 1.5 \times 10^{11}}{6.67 \times 10^9}\right)\right]^{1/2} = 633\,\mathrm{km\,s^{-1}}$$

The fractional change from 20 to 1 000 AU $= \frac{633}{494} = 1.28$, i.e., about a 30 percent increase.

A constant mass loss rate implies

$$nvr^2 = \text{constant} = n_0 v_0 r_0^2$$
$$\text{Momentum flux} = P = (nm_p v)v = m_p n_0 v_0 r_0^2 v/r^2.$$
$$v_0 \simeq 2c\,[\ln(r_0/r)]^{1/2} = 353\,\mathrm{km\,s^{-1}}.$$
$$n_0 = 3 \times 10^6\,\mathrm{m^{-3}}, \quad v \simeq (494 + 633)/2 = 564\,\mathrm{km\,s^{-1}}.$$

Therefore $P \sim 1.6 \times 10^{-27} \times 3 \times 10^6 \times 353 \times 10^3 \times 564 \times 10^3 \times (r_0/r)^2 \sim 10^{-9}/(r/r_0)^2$ Pa.

Take $P_{\mathrm{ISM}} = 2n_{\mathrm{ISM}}kT_{\mathrm{ISM}} = 2 \times 5 \times 10^5 \times 1.38 \times 10^{-23} \times 8\,000 \simeq 1.1 \times 10^{-13}$ Pa. Equating to P we get $r/r_0 = 95$, i.e., $r \simeq 95$ AU.

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11) (Exam'98) Describe how stellar winds can be driven by

(a) gas pressure from a hot corona [4]

(b) radiation pressure. [4]

How do the stellar mass loss rates compare for these two mechanisms? [2]

Neglecting photon pressure, show that a star cannot maintain a hydrostatic atmosphere unless the atmospheric temperature decreases with radial distance, r, from the star faster than $T(r) \propto r^{-1}$. [7]

Explain how, in principle, confinement by the interstellar medium could relax this condition. [3]

Solution to (11): solution from notes.

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12) (Exam'00) Describe how the radiation pressure on a dust particle gives rise to the concept of the Eddington Limiting Luminosity of a star, stating any assumptions you make. Briefly extend this description to line driven winds, qualitatively outlining the principles behind P Cygni profiles. [10]

Show that, for a dust-driven wind, the wind velocity profile, $v(r)$, varies as

$$v\frac{\mathrm{d}v}{\mathrm{d}r} = \frac{GM}{r^2}(\Gamma - 1),$$

at distances, r, well away from a star of mass M, where Γ is the ratio of the stellar luminosity to the Eddington luminosity. [6]

Hence derive the explicit velocity profile of the wind, and show that the terminal velocity at infinite distance is

$$v_\infty = v_{\rm esc}\sqrt{\Gamma - 1},$$

where $v_{\rm esc}$ is the escape velocity from the surface of the star. [6]

Qualitatively, how is this analysis modified at distances of just a few stellar radii form the star? [2]

An asymptotic giant branch star has a luminosity of $3 \times 10^4\, L_\odot$ and a terminal wind speed of $30\,{\rm km\,s^{-1}}$. Estimate the maximum mass loss rate from this star in $M_\odot\,{\rm yr}^{-1}$, assuming each photon emitted transfers all its momentum to a dust particle. [4]

Why might winds of this sort be confined to the envelopes of cool stars? [4]

Solution to (12):

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13) (Exam'00) Describe the structure of the outer regions of the Sun beyond the photosphere, including a careful account of the temperature profile and possible heating sources for the corona. Explain further how the magnetic field and rotation of the Sun influences the flow of the solar wind. [10]

Solution to (13):

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14) (Exam'00) State the underlying assumptions behind the Parker Model of the solar wind, and show how mass continuity relates the number density, $n(r)$, and the speed, $v(r)$, of a the wind at a radial distance r from the Sun. [5]

The Parker solutions have the form

$$\frac{v^2}{c^2} - 2\ln\frac{v}{c} = 4\ln\frac{r}{r_{\rm c}} + 4\frac{r_{\rm c}}{r} + \text{constant},$$

where c is the sound speed and $r_{\rm c}$ the critical radius. Sketch the plots of v/c versus $r/r_{\rm c}$ for various values of the constant in the equation, and identify the value of the constant for the critical solution, applicable to the solar wind. [3]

What are the important properties of this solution? [3]

Solution to (14):

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15) (Exam'00) Distinguish between pressure driven stellar winds, such as the solar wind, and line driven winds seen in high mass-loss rate stars. [6]

The Sun loses mass at a rate of about $1.8 \times 10^9\,{\rm kg\,s^{-1}}$. Given that the mean solar wind speed at 1 AU is $400\,{\rm km\,s^{-1}}$, estimate the ratio of the solar wind ram pressure to the photon pressure on the surface of the Moon facing the Sun. Why would the value on the surface of the Earth be different? [8]

By considering the wavelengths of light scattered out of and into the line-of-sight to a star by its stellar wind, carefully explain how a spherically symmetric line driven wind with strong resonant scattering will show the line with a P-Cygni profile. Include in your account how the line profile is affected by the maximum expansion speed of the wind, v_∞. You may assume that the wind is optically thick at its resonant frequency and that the wind speed gradually increases with radial distance to its limiting value. [12]

Deduce how the profile would be affected by

(a) The finite size of the star [2]

(b) The decreasing optical thickness of the outer wind. [2]

Solution to (15):

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16) (Class Test'00) Show that, in certain circumstances, the force due to radiation pressure on a particle a distance r from a star of luminosity L is

$$F = \frac{L\sigma}{4\pi^2 c},$$

where c is the speed of light and σ is the effective cross section of the particle to radiation. Make all your assumptions clear. [4]

Using this model, determine the critical luminosity, L_c, of a star for dust grains of mass m, and describe the motion of these grains for $L > L_\mathrm{c}$ and $L < L_\mathrm{c}$. [4]

Qualitatively, how is their behaviour modified at distances comparable with the stellar radius? [2]

Solution to (16):

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17) (Exam'02) How do we know that the Sun has

(a) a corona, at a temperature of about 10^6 K? [2]

(b) a wind, with a speed of about $350\,\mathrm{km\,s^{-1}}$? [2]

Neglecting radiation pressure, show the temperature, T, and electron number density, n, of a perfect gas of ionised hydrogen in (spherically symmetric) hydrostatic equilibrium around a star of mass M obeys

$$\frac{\mathrm{d}}{\mathrm{d}r}(2nkT) = -\frac{GMm_\mathrm{p}n}{r^2},$$

where r is the radial distance from the centre of the star, m_p is the mass of a proton, k is Boltzmann's constant and G the gravitational constant. [5]

Go on to show that, for an isolated star, the temperature must decrease with radius faster than $1/r$ for equilibrium to be possible. [6]

In the Chapman model of the Sun's corona, a slower fall-off ($T \propto r^{-2/7}$) is predicted. Outline the distinguishing physical feature of this model. [3]

Show that, close in to the Sun ($r \simeq r_0$, where r_0 is the solar radius), the pressure in such an atmosphere falls off exponentially, but far from the Sun it approaches a limiting value of

$$P = P_0 \exp\left(-\frac{7GMm_\mathrm{p}}{10kT_0r_0}\right),$$

where P_0 and T_0 are the pressure and temperature a distance r_0 from the Sun. [12]

Solution to (17):

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18) HARD Above its maximum value at a temperature T_0, the radiative loss function $f(T)$ dominant in the upper chromosphere of the Sun can be approximated by

$$f(T) = f_0\left(\frac{T_0}{T} + \frac{T}{\alpha T_0}\right).$$

A plasma in this region has a fixed number density, n, and is in thermal equilibrium at a temperature just below T_0. It suffers a radiative loss per unit volume $n^2 f(T_0)$, balancing a heat input of Cn per unit volume. Show that if this equilibrium is disturbed to a slightly higher temperature, the eventual stable equilibrium temperature is αT_0.

By considering the energy balance equation in the plasma,

$$2nk\frac{\mathrm{d}T}{\mathrm{d}t} = Cn - n^2 f(T)$$

show also that the time taken to reach this new equilibrium temperature (with steady heating) is

$$t = \frac{2\alpha k T_0}{n f_0}\int_1^{\alpha}\frac{y\ \mathrm{d}y}{(y-1)(\alpha-y)}\,.$$

Solution to (18): The radiative transfer function, $f(T)$, has the same value at T_0 and the higher temperature, T. It is in stable equilibrium at T. i.e.,

$$f_0\left(\frac{T_0}{T}+\frac{T}{\alpha T_0}\right) = f_0\left(1+\frac{1}{\alpha}\right)$$

$$T^2 - T_0(\alpha+1)T + \alpha T_0^2 = 0$$

$$T = \alpha T_0, \quad T_0.$$

Considering the energy balance:

$$2nkT\frac{\mathrm{d}T}{\mathrm{d}t} = Cn - n^2 f(T) = Cn - n^2 f_0\left(\frac{T_0}{T}+\frac{T}{\alpha T_0}\right).$$

$$\text{At } T_0,\quad 0 = Cn - n^2 f_0(1+1/\alpha),$$

$$\text{so that}\quad C = n f_0(1+1/\alpha).$$

$$\text{i.e.,}\quad \frac{\mathrm{d}T}{\mathrm{d}t} = \frac{n f_0}{2k\alpha}(\alpha+1) - \frac{n f_0}{2k}\left(\frac{T_0}{T}+\frac{T}{\alpha T_0}\right).$$

$$\text{Putting}\quad a = \frac{n f_0(\alpha+1)}{2k\alpha};\quad b = \frac{n f_0}{2k};\quad y = \frac{T}{T_0}$$

$$\text{We get}\quad T_0\frac{\mathrm{d}y}{\mathrm{d}t} = a - b\left(\frac{1}{y}+\frac{y}{\alpha}\right)$$

$$t = T_0\int_1^{\alpha}\frac{y\ \mathrm{d}y}{ay - b - by^2/\alpha}.$$

$$\text{Using}\quad a\alpha/b = \alpha+1$$

$$\text{we get}\quad t = \frac{T_0\alpha 2k}{n f_0}\int_1^{\alpha}\frac{y\ \mathrm{d}y}{(1+\alpha)y-\alpha-y^2} = \frac{2\alpha k T_0}{n f_0}\int_1^{\alpha}\frac{y\ \mathrm{d}y}{(y-1)(\alpha-y)}.$$

G.W.
2012