The radiometer equation – a more careful look

In the lectures we derive the equation

$$\mathrm{SNR} = \frac{T_\mathrm{A}}{T_\mathrm{sys}}\,(\mathit{\Delta}\nu\tau)^{1/2}\,, \tag{1}$$

sometimes called the *radiometer equation*, to determine the signal-to-noise ratio from a source generating an antenna temperature T_A using a telescope with a system temperature T_sys, a bandwidth $\mathit{\Delta}\nu$ and an integration time τ. Although this result is rigorously correct we made a couple of hand-wavy approximations along the way, so here we'll do it more carefully.

A radio source (and any other contributing noise source) will generate a randomly fluctuating voltage $u(t)$ at the antenna which we will assume has a gaussian probability distribution with a standard deviation σ and a mean of zero, so that the probability density for u is

$$p(u) = \frac{1}{(2\pi)^{1/2}\sigma}\exp\left(\frac{-u^2}{2\sigma^2}\right). \tag{2}$$

Our radio telescope system generates a signal $V(t)$, proportional to $[u(t)]^2$, which we use as our estimator for the noise power. If the signal from the source is the only noise present, then the mean of V is proportional to the source's flux density. If there are other noise sources present then only part of V will be from the source. The pdf of V is one-sided ($V \geq 0$), and is related to $p(u)$ by

$$p(V)\,|\mathrm{d}V| = 2p(u)\,|\mathrm{d}u| \quad (V \geq 0). \tag{3}$$

Using

$$\mathrm{d}V = 2u\,\mathrm{d}u, \tag{4}$$

we can write

$$p(V) = \frac{V^{-1/2}}{(2\pi)^{1/2}\sigma}\exp\left(\frac{-V^2}{2\sigma^2}\right). \tag{5}$$

This is known as a *chisquared distribution with one degree of freedom*, and has a mean of

$$\langle V\rangle = \int_0^\infty V\,p(V)\,\mathrm{d}V = \sigma^2, \tag{6}$$

and a variance of

$$\begin{aligned}\mathrm{var}[V] &= \langle V^2\rangle - \langle V\rangle^2 && (7)\\ &= \int_0^\infty V^2 p(V)\,\mathrm{d}V - \sigma^4 && (8)\\ &= 2\sigma^4. && (9)\end{aligned}$$

The standard deviation of V is therefore $\sqrt{2}\sigma^2$, or $\sqrt{2}\langle V\rangle$. If the random signal is the only 'noise' in the system, then the signal-to-noise ratio is simply

$$\mathrm{SNR} = \frac{\langle V\rangle}{[\mathrm{var}(V)]^{1/2}} = \frac{1}{\sqrt{2}}. \tag{10}$$

In the lectures we use a plausibility argument to say that this signal-to-noise ratio is '$\simeq 1$', but the result above is more rigorous. If other noise sources are present things are a little more complicated, but the basic argument still holds. The signal will now be represented by the power from just the source, proportional to the antenna temperature T_A, and the total noise power by the system temperature T_sys, so that

$$\mathrm{SNR} = \frac{T_\mathrm{A}}{T_\mathrm{sys}\sqrt{2}}. \tag{11}$$

We can now consider how to improve this signal-to-noise ratio by integrating (averaging) the sample values of V over some interval of time τ. Let there be N samples of V (or u) in time τ. If the samples are statistically independent, and $N \gg 1$, then (by definition) the standard deviation of the average is $\sqrt{N}$ less than that of a sample signal-to-noise ratio, i.e.,

$$\mathrm{SNR} = \frac{T_\mathrm{A}}{T_\mathrm{sys}\sqrt{2}}\sqrt{N}. \tag{12}$$

Samples u_i and u_j will be independent if

$$\langle u_i u_j\rangle = \langle u^2\rangle\delta_{ij}, \tag{13}$$

i.e., if the autocorrelation of u is a delta function at the origin. This corresponds (by the Wiener–Khinchin theorem) to a white ('flat') power spectrum for the noise. The Nyquist sampling theorem tells us that a sampling interval of τ/N corresponds to a bandwidth of $\mathit{\Delta}\nu$ where

$$\mathit{\Delta}\nu = \frac{1}{2}\frac{N}{\tau}, \tag{14}$$

so

$$N = 2\mathit{\Delta}\nu\tau, \tag{15}$$

(in the lectures we say that '$N \simeq \mathit{\Delta}\nu\tau$'), and therefore our final signal-to-noise ratio is

$$\mathrm{SNR} = \frac{T_\mathrm{A}}{T_\mathrm{sys}2^{1/2}}(2\mathit{\Delta}\nu\tau)^{1/2} = \frac{T_\mathrm{A}}{T_\mathrm{sys}}\,(\mathit{\Delta}\nu\tau)^{1/2}\,. \tag{16}$$

Note that the two factors of $\sqrt{2}$ introduced by our more careful analysis cancel out, so the precise and 'approximate' results are the same.

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