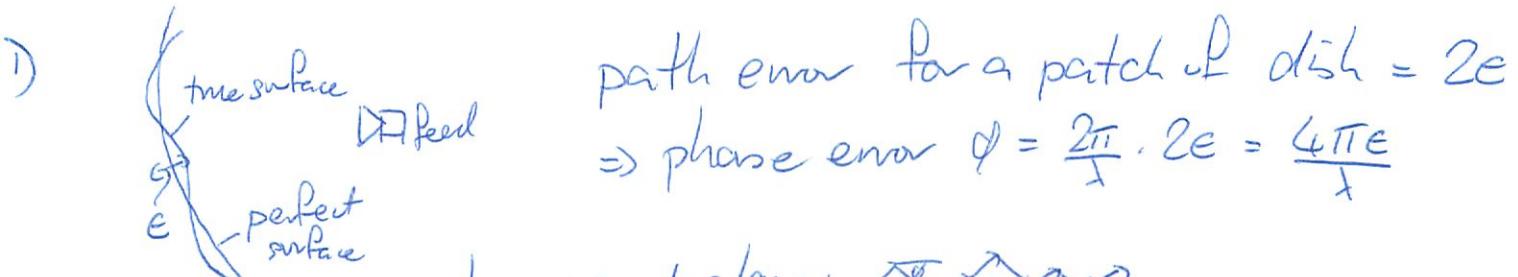


Instrumentation II - Sketch answers to question sheet 2.



In argand plane:

overall voltage response is down by $\sim \cos \phi$

$$\Rightarrow \text{power response down by } Q = \cos^2 \phi = \underline{\cos^2(4\pi e/\lambda)}$$

$$\text{If } \eta_s = 0.8 \quad (\lambda = 0.06 \text{ m}) \Rightarrow e = 2.21 \times 10^{-3} \text{ m}$$

$$\text{At } 15 \text{ GHz, } \lambda = 0.02 \text{ m} \Rightarrow Q_{15} = \cos^2 \left(\frac{4\pi \times 2.21 \times 10^{-3}}{0.05} \right) = 0.032$$

2) Aperture A^y

$$= A(x) \otimes \left(\begin{array}{c} \xrightarrow{20a} \\ \text{B}(x) \end{array} \right) \times \left(\begin{array}{c} \xleftarrow{-\infty} \\ \text{C}(x) \end{array} \right)$$

Amplitude pattern for $A(x)$: $V_A(\theta) \propto \int_{-\infty}^{\infty} A(x) e^{iksx} dx$, where $s = \sin \theta$

$$\therefore V_A(\theta) \propto \frac{2}{ik_s} \sin \frac{kDs}{2}$$

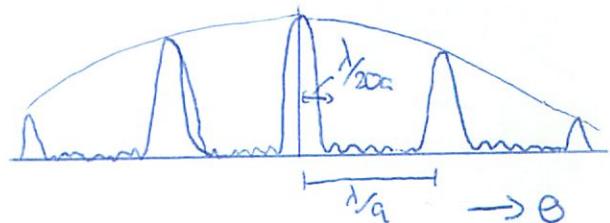
Define: $\text{sinc} s = \frac{\sin \pi s}{\pi s}$; $\Rightarrow V_A(\theta) \propto \frac{D}{i} \text{sinc} \frac{Ds}{\lambda}$

Similarly, $V_B(\theta) \propto \frac{20a}{i} \text{sinc} \frac{20as}{\lambda}$

$$\text{Also, } V_C(\theta) = \int_{-\infty}^{\infty} C(x) e^{iksx} dx = \sum_{n=-\infty}^{\infty} \delta(s - \frac{n\lambda}{a})$$

$$V_{\text{tot}}(\theta) \propto V_A(\theta) \times (V_B(\theta) \otimes V_C(\theta)); \quad P(\theta) \propto |V_{\text{tot}}(\theta)|^2; \quad P(0) = 1$$

$$\Rightarrow P(\theta) = \text{sinc}^2 \frac{Ds}{\lambda} \left[\sum_{n=-\infty}^{\infty} \text{sinc} \frac{20a(s - \frac{n\lambda}{a})}{\lambda} \right]^2$$



3) Total noise power per unit bandwidth = kT_{sys} . T_{sys} includes contributions from all noise sources: $\bar{T}_{\text{sys}} = T_{\text{antenna}} + T_{\text{electronics}} + \dots$

For an unpolarised source, flux density S :

Power per unit bandwidth from source = $\frac{1}{2} AeS$

\Rightarrow instantaneous SNR = $\frac{Y_2 AeS}{k\bar{T}_{\text{sys}}}$. For \approx bw. of ΔV and an integration

time of τ , number of independent samples $N = \Delta V \tau$. SNR \propto by $\sqrt{N} \Rightarrow$

$$\text{SNR} = \frac{Y_2 AeS}{k\bar{T}_{\text{sys}}} \sqrt{\Delta V \tau}$$

$$\text{Set SNR} = 1 \Rightarrow S_{\min} = \frac{2k\bar{T}_{\text{sys}}}{Ae\sqrt{\Delta V \tau}}$$

$\boxed{\text{SNR} \propto Ae\sqrt{\Delta V}}$, so better to increase Ae than to increase ΔV .

4) Argument as in (3) above. Signal = $k\bar{D}\bar{T}$; $N = \Delta V \tau$

$$\text{noise} = k\bar{T}_{\text{sys}}$$

$$\Rightarrow \text{SNR} = \frac{\bar{D}\bar{T}}{\bar{T}_{\text{sys}}} \sqrt{\Delta V \tau}$$

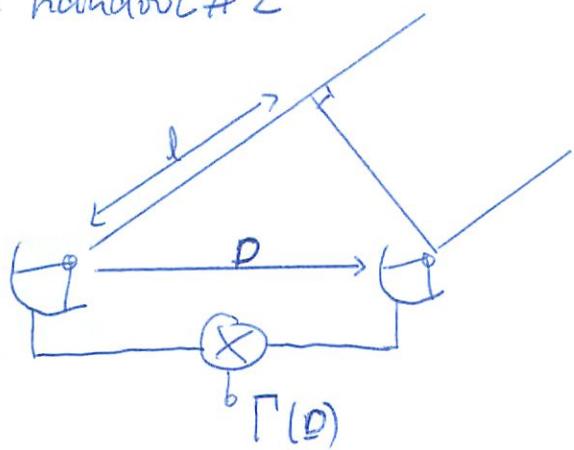
$$\text{If } 10 = \frac{3 \times 10^{-5}}{250} \sqrt{(8 \times 10^8 \tau)} \Rightarrow \tau = 8.68 \times 10^6 \text{ s} = \underline{100 \text{ days}}$$

5)

$$S_{\min} = \frac{2k\bar{T}_{\text{sys}}}{Ae\sqrt{\Delta V \tau}}$$

$$\left. \begin{array}{l} S = 1.1 \times 10^{-26} \text{ W Hz}^{-1} \text{ m}^{-2} \\ \bar{T}_{\text{sys}} = 180 \text{ K} \\ \Delta V = 8 \times 10^6 \text{ Hz} \\ \tau = 600 \text{ s} \end{array} \right\} \Rightarrow \underline{Ae = 6.52 \text{ m}^2}$$

6) See handout #2



The fringe rate is the rotation rate of the complex fringe visibility $\Gamma(D)$ in the angular plane. It equals the rate of change of ℓ , measured in λ .

$$\phi = \frac{2\pi \ell}{\lambda} = \frac{2\pi D}{\lambda} \cos \delta \sin H \quad \text{if the interferometer is E-W.}$$

$$\text{Fringe rate} = \frac{d\phi}{dt} = \frac{2\pi}{\lambda} D \cos \delta \cos H \frac{dH}{dt}.$$

$$\text{At the meridian, } H=0, \text{ so } Fr = \frac{2\pi}{\lambda} \cos \delta \frac{\dot{H}}{C} \approx \frac{2\pi}{86400} \text{ radians sec}^{-1}$$

If $D = 200m, \lambda = 2m$

$$\text{at } \delta = 30^\circ, Fr = 0.0396 \text{ rad s}^{-1} = \underline{6.3 \text{ mHz}}$$

$$\text{at } \delta = 80^\circ, Fr = 7.93 \times 10^{-3} \text{ " } = \underline{1.26 \text{ mHz}}$$

If the baseline is N-S $D = (0, D, 0)$

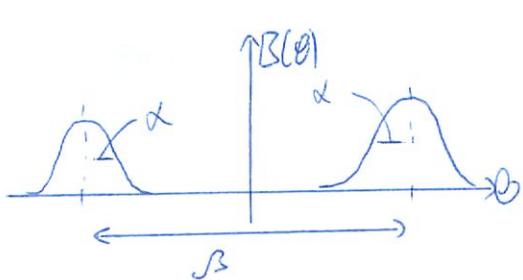
$$D \cdot \hat{\theta} = \cos H \cos \delta$$

$$\phi = \frac{2\pi}{\lambda} \cos \delta \cos H$$

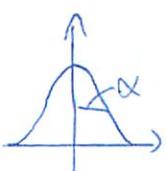
$$\dot{\phi} = \frac{2\pi}{\lambda} \cos \delta \sin H \dot{H}$$

so that at $H=0, Fr=0$

8) The model for Cygnus A looks like



$$\text{i.e. } B(\theta) \propto \frac{1}{\beta} \rightarrow \otimes$$



8 cont. $\Gamma(r) \propto FT[B(\theta)]$ where $FT[A(\theta)] = \int A(\theta) e^{ikr\theta} d\theta$

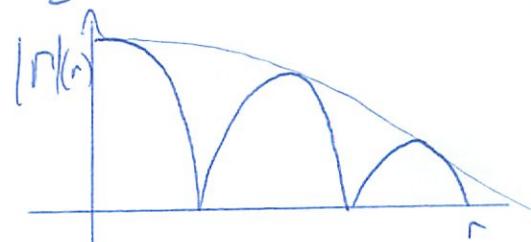
$$FT[\overrightarrow{11}] \propto e^{-ik\frac{\beta}{2}r} + e^{ik\frac{\beta}{2}r} \propto \cos k\frac{\beta r}{2}$$

$$FT[e^{-\theta/\lambda^2}] \propto e^{-\pi^2 \alpha^2 r^2 / \lambda^2}$$

$$\text{so } \Gamma(r) \propto FT[\overrightarrow{11} \otimes \overrightarrow{A}] = FT[\overrightarrow{11}] \times FT[\overrightarrow{A}] \\ = e^{ik\frac{\beta r}{2}} \cdot e^{-\pi^2 \alpha^2 r^2 / \lambda^2}$$

$$\Gamma(0) = 1$$

$$\Rightarrow \underline{\Gamma(r)} = e^{-\pi^2 \alpha^2 r^2 / \lambda^2} \cos \pi r \beta / \lambda$$

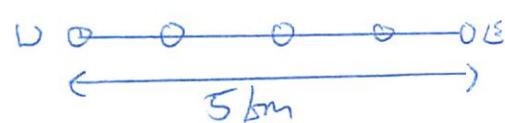


Γ first drops to zero when $\pi r \beta = \frac{\pi}{2}$

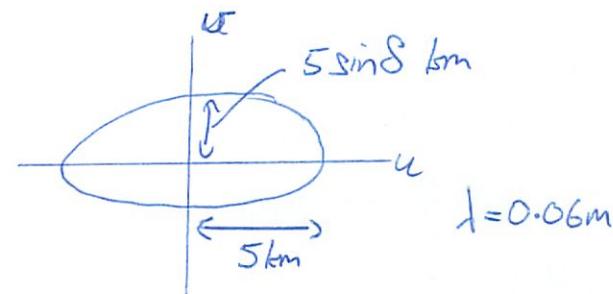
$$\begin{cases} \beta = 5.33 \times 10^{-4} \\ \lambda = 7.89 \text{ m} \end{cases} \Rightarrow \underline{r = 7.4 \text{ km}}$$

9. Primary beam = beam of single dish

Synthesised beam = effective beam of synthesised aperture (i.e. the 'resolution' of the map)



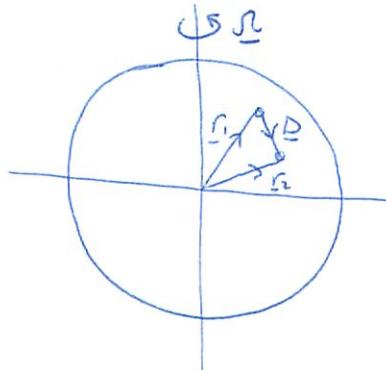
in (u, v) plane
[aperture plane]



$$\text{resolution in RA } (\parallel u) = \frac{\lambda}{D} = 1.2 \times 10^{-5} = \underline{2.5 \text{ arcsec}}$$

$$\text{resolution in dec } (\parallel v) = \frac{\lambda}{D \sin \delta} = 3.5 \times 10^{-5} = \underline{7.2 \text{ arcsec}}$$

10.



In general, let the ends of the baseline be at r_1, r_2 wrt the Earth centre. Source @ $\hat{\theta}$.

$$\Delta = r_2 - r_1. \text{ Speeds of ends: } v_1 = r_1 \times \underline{\Omega} \\ v_2 = r_2 \times \underline{\Omega}$$

$$\text{Doppler shift: } \Delta v = \frac{v}{c} \underline{v} \cdot \hat{\theta} = \frac{1}{\lambda} \underline{v} \cdot \hat{\theta}$$

10 cont...

(5)

differential doppler shift =

$$\Delta V_1 - \Delta V_2 = \frac{1}{\lambda} (\underline{v}_1 - \underline{v}_2) \cdot \hat{\underline{\Omega}} = \frac{1}{\lambda} (\underline{\Omega} \times \underline{D}) \cdot \hat{\underline{\Omega}}$$

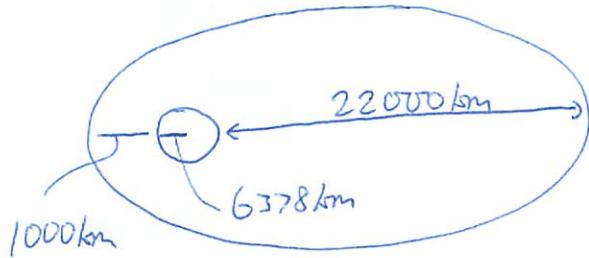
For an E-W interferometer, $\underline{D} = (D, 0, 0)$

$$\underline{\Omega} = (0, 0, \Omega)$$

$$\hat{\underline{\Omega}} = (\sin H \cos \delta, \cos H \cos \delta, \sin \delta)$$

$$\therefore \Delta V_1 - \Delta V_2 = \frac{D}{\lambda} \cos H \cos \delta \Omega \equiv \frac{D}{\lambda} \cos \delta \cos H \dot{H} = \text{fringe rate.}$$

11



$$r_a = 28378 \text{ km}$$

$$r_p = 7378 \text{ km}$$

a) Max resolution = $\frac{\lambda}{D_{\max}} = \frac{0.0136}{34756 \times 10^3} = 3.913 \times 10^{-10} = 81 \mu\text{arcsec}$

b) $\tau \approx \frac{1}{\Delta V} = 62.5 \text{ ns}$

c) $m v_p r_p = m v_a r_a ; \frac{1}{2} m v_p^2 - \frac{GMm}{r_p} = \frac{1}{2} m v_a^2 - \frac{GMm}{r_a}$

$$\Rightarrow v_p^2 = \frac{GM r_a}{r_p(r_a + r_p)} = R_{\text{Earth}} g \frac{r_a}{r_p(r_a + r_p)} \Rightarrow v_p = 6527 \text{ m/s}$$

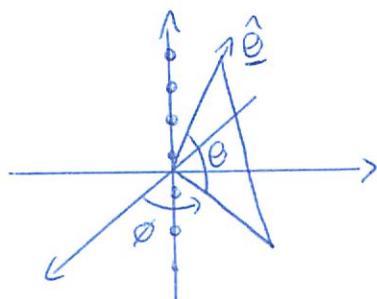
doppler shift $\Delta V = \frac{v}{c} v = \frac{v}{\lambda} \cdot \lambda = 18.75 \text{ m} \Rightarrow \Delta V = 479 \text{ kHz}$
 $(= \text{Frz})$

long baselines \Rightarrow imaging high spatial frequencies. Need lots of flux on small angular scale. Surface brightness $B = \frac{S \propto \text{flux}}{A \propto \text{solid angle}}$, so need high B .

Mat. scale-size $\approx \frac{\lambda}{D_{\min}} = \frac{0.0136}{1000 \text{ km}} = 1.36 \times 10^{-8} = 2.81 \mu\text{arcsec}$

12.

a)



In general, antenna voltage pattern is

$$V(\theta, \phi) \propto \int A(r) e^{ikr \cdot \hat{e}} dr$$

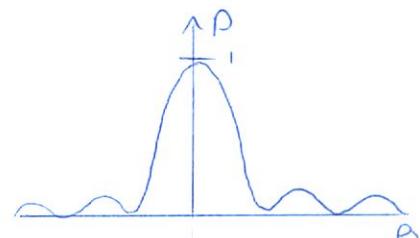
$$\text{here } \propto \sum_{n=1}^{16} \exp \left[\frac{2\pi i}{\lambda} \cdot \frac{n\lambda}{2} \sin \theta \right]$$

$$P_v t D = \pi \sin \theta \Rightarrow V \propto \sum_{n=1}^{16} e^{inD}$$

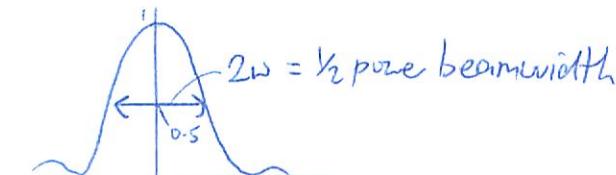
$$16 \text{ is a geometric progression: } V \propto \frac{e^{iD} (1 - e^{i16D})}{(1 - e^{iD})}$$

$$\text{So } P(\theta, \phi) = |V|^2 \propto \frac{\sin^2 [8\pi \sin \theta]}{\sin^2 [\pi/2 \sin \theta]}$$

$$P(0,0) = 1 \Rightarrow P(\theta, \phi) = \frac{1}{16^2} \frac{\sin^2 [8\pi \sin \theta]}{\sin^2 [\pi/2 \sin \theta]}$$



b)



$$P(\theta) = \frac{1}{2} = \frac{1}{16^2} \frac{\sin^2 [8\pi \sin \omega]}{\sin^2 [\pi/2 \sin \omega]}$$

$$\begin{aligned} \text{Use Maple, or substitute the answer} &\Rightarrow \omega = 0.05549 \text{ rad} \\ &\Rightarrow \underline{2\omega = 6^\circ 22'} \end{aligned}$$

$$\text{c) First subsid. maxima when } 8\pi \sin \theta = \frac{3\pi}{2}$$

$$\therefore P(\theta, \phi) = \frac{1}{16^2} \cdot \frac{1}{\sin^2 (3\pi/2)} = 0.0463$$

$$dB = 10 \log P \Rightarrow \underline{-13.3 dB}$$

$$\begin{aligned} \text{d) } S_A &= \int P(\theta, \phi) d\Omega = 2 \cdot 2\pi \int_{\theta=0}^{\pi/2} \frac{1}{16^2} \frac{\sin^2 (8\pi \sin \theta)}{\sin^2 (\pi/2 \sin \theta)} \cos \theta d\theta \\ &= \frac{\pi}{64} \int_0^1 \frac{\sin^2 8\pi r}{\sin^2 \pi r/2} dr \end{aligned}$$

Solve this by using earlier notation:

$$\begin{aligned}
 \frac{\sin^2(8\pi x_2)}{\sin^2(\pi x_2)} &= \left| e^{i\Delta} + e^{2i\Delta} + e^{3i\Delta} + \dots + e^{16i\Delta} \right|^2 \quad (\Delta = \pi x_2) \\
 &= |1 + e^{-i\Delta} + e^{-2i\Delta} + \dots + e^{-15i\Delta} \\
 &\quad + e^{i\Delta} + 1 + e^{-i\Delta} + \dots + e^{-14i\Delta} \\
 &\quad + e^{2i\Delta} + e^{i\Delta} + 1 + \dots + e^{-13i\Delta} \\
 &\quad + \dots \\
 &\quad + e^{15i\Delta} + e^{14i\Delta} + e^{13i\Delta} + \dots + 1| \\
 &= 16 + 15(e^{i\Delta} + e^{-i\Delta}) + 14(e^{2i\Delta} + e^{-2i\Delta}) + \dots + (e^{15i\Delta} + e^{-15i\Delta}) \\
 &= 16 + 30 \cos \pi x_2 + 28 \cos 2\pi x_2 + \dots + 2 \cos 15\pi x_2
 \end{aligned}$$

$$\begin{aligned}
 \text{So } S_A &= \frac{\pi}{64} \int_0^1 (16 + 30 \cos \pi x_2 + 28 \cos 2\pi x_2 + \dots + 2 \cos 15\pi x_2) dx_2 \\
 &= \underline{\underline{\frac{\pi}{4}}}
 \end{aligned}$$

e) [gulp!] Beam efficiency $\eta = \frac{S_m}{S_A}$ ^{main lobe}, 1st zero at $\sin \theta = \frac{1}{8}$

$$\begin{aligned}
 \text{So } S_m &= \frac{4\pi}{16} \int_0^{\sin^{-1} \frac{1}{8}} \frac{\sin^2(8\pi \sin \theta)}{\sin^2(\pi x_2 \sin \theta)} \cos \theta d\theta \\
 &= \frac{\pi}{64} \int_0^{\frac{1}{8}} \frac{\sin^2(8\pi x_2)}{\sin^2(\pi x_2)} dx_2 \\
 &= \frac{\pi}{64} \left(\frac{16}{8} + \frac{30}{\pi} \left[\sin \pi x_2 \right]_0^{\frac{1}{8}} + \frac{28}{2\pi} \left[\sin 2\pi x_2 \right]_0^{\frac{1}{8}} \right. \\
 &\quad \left. + \dots + \frac{2}{15\pi} \left[\sin 15\pi x_2 \right]_0^{\frac{1}{8}} \right)
 \end{aligned}$$

$$i.e. S_m = \frac{1}{64} \left[2\pi + 30 \sin \frac{\pi}{8} + \frac{28}{2} \sin \frac{2\pi}{8} + \frac{26}{3} \sin \frac{3\pi}{8} + \frac{24}{4} \sin \frac{4\pi}{8} \right. \\ \left. + \frac{22}{5} \sin \frac{5\pi}{8} + \frac{20}{6} \sin \frac{6\pi}{8} + \frac{18}{7} \sin \frac{7\pi}{8} + \frac{16}{8} \sin \pi \right. \\ \left. + \frac{14}{9} \sin \frac{9\pi}{8} + \frac{12}{10} \sin \frac{10\pi}{8} + \frac{10}{11} \sin \frac{11\pi}{8} + \frac{8}{12} \sin \frac{12\pi}{8} \right. \\ \left. + \frac{6}{13} \sin \frac{13\pi}{8} + \frac{4}{14} \sin \frac{14\pi}{8} + \frac{2}{15} \sin \frac{15\pi}{8} \right]$$

$$= \frac{1}{64} \left[2\pi + \frac{9728}{315} \sin \frac{\pi}{8} + \frac{1664}{105} \sin \frac{\pi}{4} + \frac{25088}{2145} \sin \frac{3\pi}{8} + \frac{16}{3} \right] \\ \text{where } \left(\frac{\sqrt{2}-1}{2\sqrt{2}} \right)^k$$

$$= 0.7101$$

$$\Rightarrow \eta = \frac{0.7101}{\pi/4} = \underline{0.90}$$

f) effective aperture A_e : $A_e = \frac{\lambda^2}{S_A} = \frac{4\lambda^2}{\pi} = \underline{1.27\lambda^2}$