EXAM WITH SOLUTIONS

summer
time

ASTRONOMY DEGREE EXAMINATION FOR BSC (DESIGNATED), BSC (HONOURS), MSCI (HONOURS) AND MSC DIETS

Astronomy 345HM past papers

AB01H

[ASTRO4010, ASTRO3011]

Instrumentation: (Optical and) Radio

This is an assembly of past exam questions in IOR1 typeset in exam format to get you used to it!

Answer each question in a separate booklet

Electronic devices (including calculators) with a facility for either textual storage or display, or for graphical display, are excluded from use in examinations.

Approximate marks are indicated in brackets as a guide for candidates.

speed of light in vacuum	c	2.997 924 58	$\times 10^{8}$ m s^{-1}
permeability of vacuum	μ_0	4π	$\times 10^{-7}$ H m^{-1}
permittivity of vacuum	ϵ_0	8.854 187 817 . . .	$\times 10^{-12}$ F m^{-1}
constant of gravitation	G	6.673 84(80)	$\times 10^{-11}$ m^{3} kg^{-1} s^{-2}
Planck constant	h	6.626 069 57(29)	$\times 10^{-34}$ J s
$h/(2\pi)$	$\hbar$	1.054 571 726(47)	$\times 10^{-34}$ J s
elementary charge	e	1.602 176 565(35)	$\times 10^{-19}$ C
electron volt	eV	1.602 176 565(35)	$\times 10^{-19}$ J
electron mass	m_{e}	9.109 382 91(40)	$\times 10^{-31}$ kg
proton mass	m_{p}	1.672 621 777(74)	$\times 10^{-27}$ kg
unified atomic mass unit	u	1.660 538 921(73)	$\times 10^{-27}$ kg
fine-structure constant	α	7.297 352 5698(24)	$\times 10^{-3}$
Rydberg constant	R_∞	1.097 373 156 853 9(55)	$\times 10^{7}$ m^{-1}
Avogadro constant	N_{A}	6.022 141 29(27)	$\times 10^{23}$ mol^{-1}
molar gas constant	R	8.314 462 1(75)	J mol^{-1} K^{-1}
Boltzmann constant	k_{B}	1.380 648 8(13)	$\times 10^{-23}$ J K^{-1}
Stefan–Boltzmann constant	σ	5.670 373(21)	$\times 10^{-8}$ W m^{-2} K^{-4}
Bohr magneton	μ_{B}	9.274 008 99(37)	$\times 10^{-24}$ J T^{-1}
jansky	Jy	1	$\times 10^{-26}$ W m^{-2} Hz^{-1}
astronomical unit	au	1.495 978 707	$\times 10^{11}$ m
parsec	pc	3.085 677 6	$\times 10^{16}$ m
light-year	ly	9.460 730 472 . . .	$\times 10^{15}$ m
Sun's mass	$M_\odot$	1.988 55(24)	$\times 10^{30}$ kg
Sun's equatorial radius	$R_\odot$	6.963 42(65)	$\times 10^{8}$ m
Sun's luminosity	$L_\odot$	3.839(5)	$\times 10^{26}$ W
Earth's mass	$M_\oplus$	5.972 58(71)	$\times 10^{24}$ kg
Earth's equatorial radius	$R_\oplus$	6.378 1366(1)	$\times 10^{6}$ m

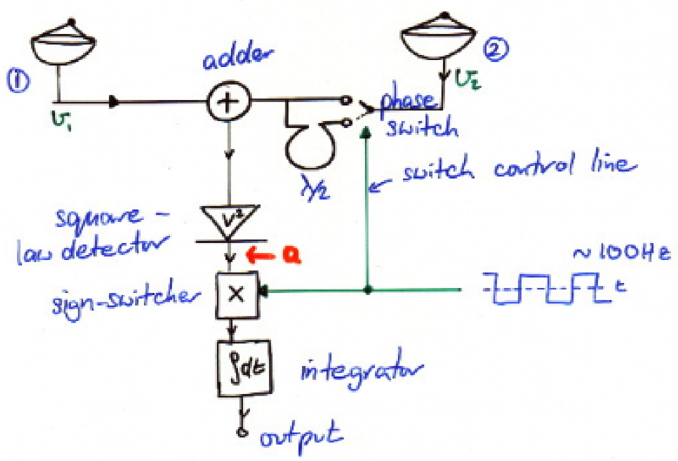
Paper continued over

1 (a) Signal multiplication (or correlation) plays an important role in radio interferometry and imaging. Without going into detail, state the relationship between the correlation of signals received at two separated antennas and the sky brightness distribution of the source that generated those signals. **[4]**

Solution: The fundamental relationship here is the van Cittert-Zernike Theorem: "The complex fringe visibility is the Fourier transform of the normalised sky brightness". So there is a direct relationship between these two, but how does the complex fringe visibility relate to the correlation between two signals? Although defined in terms of hypothetical fringes generated by the interference of signals from the two antennas projected onto a distant screen, the complex fringe visibility is mathematically equivalent to the correlation coefficient between the two signals, so it is proportional to the complex mean product of the two signals. Therefore, this correlation value is proportional to the Fourier component on the sky corresponding to the (reciprocal) baseline vector.

(b) With the aid of a diagram, explain how a phase-switched interferometer generates a signal proportional to the product of the signals from two antennas. **[4]**

The phase-switched interferometer



• form $(v_1+v_2)^2 = v_1^2 + v_2^2 + 2v_1v_2$

$(v_1-v_2)^2 = v_1^2 + v_2^2 - 2v_1v_2$

Alternately at **a**. Switch signs of the two products synchronously with the phase switch

$\Rightarrow$ output $= \langle [v_1^2+v_2^2+2v_1v_2] - [v_1^2+v_2^2-2v_1v_2] \rangle$

Solution: $= 4\langle v_1 v_2 \rangle$

(c) How is correlation performed between 1-bit digital signals? **[2]**

Q 1 continued

Solution: 1-bit signals encode whether the original analogue signal was either greater than zero (1) or less than zero (0). Correlation of two such streams should generate a '1' of they have the same sign and a '0' is the signs are different. The truth table of this corresponds to an exclusive NOR gate.

[Total: 10]

2 (a) Describe the major components in the signal chain of a total power radio telescope, from antenna to measurement, explaining the purpose and necessity of each component. **[12]**

Solution: The block parts are:

i. the antenna, turning the EM wave into a voltage. A large antenna collects more EM power, either by concentrating it towards a smaller antenna at a focus or by summing signals from many antennas in phase.

ii. the preamplifier takes this weak voltage and boosts it by a factor of about 1000 so that it is strong enough for its signal-to-noise ratio (snr) to be unaffected by noise introduced by what is to come. The preamp often defines the system temperature of the telescope.

iii. a filter to define the bandwidth of the signal considered and to reject out-of-band interference.

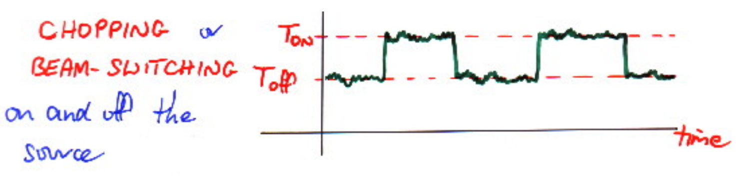
iv. a mixer and local oscillator to shift the signal frequency down to a more manageable range for further amplification and detection.

v. a square law detector to generate a voltage proportional to the power in the signal being presented to it.

vi. an integrator to average the noise fluctuations in the output of the detector and increase the snr.

(b) Distinguish between the antenna temperature and the system temperature of a radio telescope and explain why and how chopping (beam-switching) is often employed to measure antenna temperature. **[5]**

Solution: We think in terms of temperature as a measure of power per unit bandwidth. The power from the antenna itself, from both the source we are looking at with the telescope and anything else that is coming in the front-end (such as background noise or spillover) is proportional to the "antenna temperature". This plus all other sources of noise in the signal path, including noise from the preamplifier and the receiving electronics, is called the "system temperature". Chopping involves alternately pointing the telescope towards and away from the source:



The signal from the source can be estimated from

$$T_{\rm source} = T_{\rm on} - T_{\rm off}.$$

Q 2 continued

(c) The Nyquist noise theorem states that the thermally-generated noise power delivered by a resistor at a temperature T into a matched load is

$$w = k_{\mathrm{B}} T \Delta\nu.$$

where $\Delta\nu$ is the bandwidth of the noise. Explain the relevance of this to computing the signal power received from an antenna pointed at a blackbody source that fills the beam of the antenna. **[4]**

Solution: $w = k_{\mathrm{B}} T \Delta\nu$ is essentially a thermodynamic statement, telling us the exchangeable power generated by a hot resistor. If we connected this to a matched antenna in a blackbody cavity also at temperature T then no net power can flow between the cavity and the resistor (or we would contravene the second law of thermodynamics). Therefore the antenna must also be generating a power of $w = k_{\mathrm{B}} T \Delta\nu$. Note here we have put the antenna in a cavity, so that the radiation field fills its beam. We can take the antenna out of the cavity so long as its field of view is still occupied by a patch of sky at temperature T. Hence an antenna pointing at a patch of sky of brightness temperature T (in the Rayleigh Jeans limit) that also fills its beam will generate this amount of power over its bandwidth, independent of the beam solid angle or the antenna area.

(d) A radio telescope, working in the Rayleigh-Jeans limit, is used to measure the cosmic microwave background radiation (CMBR), which appears as a nearly-isotropic 2.7 K blackbody signal over the sky. There is a small dipole anisotropy in the apparent CMBR caused by our motion through it, of magnitude

$$\frac{\Delta T}{T} = \frac{v}{c} \simeq 10^{-3}.$$

Given the system temperature of the telescope is 50 K and its bandwidth is 1 GHz, estimate the time it would take to measure this anisotropy to 1 percent and suggest a practical procedure for doing so. **[6]**

Solution: The CMBR is, as stated in the question, nearly isotropic, so it definitely fills the beam of the telescope. It is therefore fair to say that the signal power is $w = k_{\mathrm{B}} T \Delta\nu$, where $\delta\nu$ is the bandwidth of 1 GHz, and T is the temperature of the CMBR that fills the beam. The number of independent measurements made in a time τ is simply $\Delta\nu\tau$, so the snr after integrating for τ is

$$\gamma = \frac{T_{\mathrm{A}}}{T_{\mathrm{sys}}} (\Delta\nu\tau)^{1/2}.$$

To measure the anisotropy to 1 percent we need to measure the temperature to 1 part in 10^5, so the snr also has to be about 10^5. This will take a time

$$\tau = \frac{1}{\Delta\nu} \left(\gamma \frac{T_{\mathrm{sys}}}{T_{\mathrm{A}}} \right)^2 = \text{=57 minutes},$$

making these measurements at two points (the maximum and minimum of the anisotropy 180 degrees apart) . Overall therefore it would take at least two hours.

Q 2 continued

(e) What systematic effects might limit the accuracy of this measurement? **[3]**

Solution: In practice this would be very difficult and the whole system would have to remain stable to 1 part in 10^5 for the whole observation and systematics would need to be kept below this between the two observations. From Earth it would be nearly impossible as atmospheric emission and absorption, spillover etc could not be corrected for to this precision. In space it is possible (and has been done!)

[Total: 30]

3 Observations of solar radio bursts are made using a receiver of bandwidth 300 kHz connected to a log-periodic antenna with a gain of 10. The receiver frequency is constantly swept from 40 to 80 MHz dwelling for 1.25 ms at each frequency to build up a spectrum of the burst.

(a) Calculate the beam solid angle of the antenna at 60 MHz (in steradians), and state whether the antenna sees the Sun (angular diameter 0.5 degrees) as a resolved or unresolved source. **[3]**

Solution: The gain of an antenna is defined as $G = 4\pi/\Omega_A$, so in this case ($G = 10$) the beam solid angle is $\Omega_A = 2\pi/5$ sr. The solid angle of the Sun is approximately

$$\Omega_\odot = \pi\left(\frac{0.5}{2}\frac{\pi}{180}\right)^2 \simeq 6\times10^{-5}\,\mathrm{sr},$$

which is 21 000 times smaller than the antenna beam, so the Sun is very much unresolved (the Sun appears somewhat bigger than this at these frequencies, but it's still unresolved!).

(b) What is the effective collecting area of the antenna at 60 MHz? **[1]**

Solution: The effective area is $A_e = \lambda^2/\Omega_A$, which at $\lambda = c/\nu = 5$ m is $125/(2\pi)\,\mathrm{m}^2 = 19.9\,\mathrm{m}^2$.

(c) A powerful burst is seen with a signal-to-noise ratio of 10 at each frequency in a single sweep. Given the system temperature of the telescope is 3000 K at 60 MHz, calculate the strength of the burst at this frequency in janskys. **[6]**

Solution: By definition of antenna temperature T_A, a telescope of effective area A_e looking at an unpolarised source of flux density S sees a power per unit bandwidth of

$$\frac{1}{2}SA_e = k_B T_A,$$

Instantaneous signal-to-noise ratio is simply T_A/T_{sys}, improved by the square root of the number of independent samples in the observing time, $\Delta\nu\tau$, so that

$$\mathrm{SNR} = \frac{T_A}{T_{sys}}\sqrt{\Delta\nu\tau} = \frac{1}{2}\frac{SA_e}{k_B T_{sys}}\sqrt{\Delta\nu\tau}.$$

The flux density this corresponds to is therefore

$$S = \mathrm{SNR}\frac{2k_B T_{sys}}{A_e\sqrt{\Delta\nu\tau}} = 2.15\times10^5\,\mathrm{Jy}$$

[Total: 10]

4 (a) A single-dish antenna of diameter d and aperture efficiency η is used to detect a radiogalaxy of surface brightness $B(\boldsymbol{s})$, where $\boldsymbol{s}$ is a unit vector defining direction on the sky. Show that the antenna temperature is

$$T_{\rm A} = \frac{1}{2k_{\rm B}}\pi(d/2)^2\eta \int B(\boldsymbol{s})P(\boldsymbol{s})\,\mathrm{d}\Omega,$$

where $P(\boldsymbol{s})$ is the antenna power pattern, $\mathrm{d}\Omega$ a differential solid angle in the direction $\boldsymbol{s}$ and $k_{\rm B}$ is the Boltzmann constant. **[5]**

Solution: By the definition of antenna temperature, for an unpolarised source of flux density S

$$\frac{1}{2}SA_{\rm e} = k_{\rm B}T_{\rm A}.$$

The apparent flux density from a source of surface brightness B, as seen by an antenna with power pattern P is

$$S = \int B(\boldsymbol{s})P(\boldsymbol{s})\,\mathrm{d}\Omega.$$

Finally, the effective area of an antenna of diameter d and aperture efficiency η is

$$A_{\rm e} = \pi(d/2)^2\eta.$$

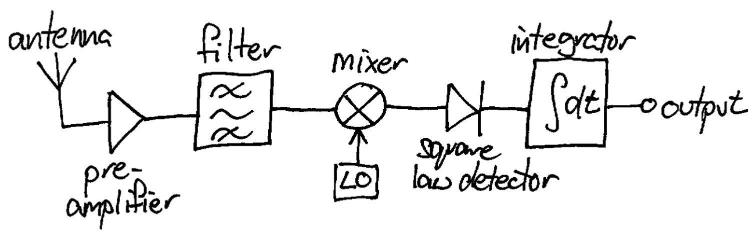
Putting these all together we get

$$T_{\rm A} = \frac{1}{2k_{\rm B}}\pi(d/2)^2\eta \int B(\boldsymbol{s})P(\boldsymbol{s})\,\mathrm{d}\Omega$$

as required.

(b) Describe, in block diagram form, how such an antenna could be used as part of a total-power radio telescope. Include in your answer the major receiver components, their purposes and their contributions to the noise in the system. **[12]**

Solution:



The blocks in the diagram are:

Antenna: This turns the EM wave into a voltage $v(t)$. This voltage has an approximately white spectrum with (by definition) a power per unit frequency interval of $k_{\rm B}T_{\rm A}$. The antenna does not contribute much noise itself, but it picks up noise sources (including the astronomy) that contribute to the overall noise level, sometimes overwhelmingly.

Q 4 continued

Preamplifier: This boosts $v(t)$ by $\sim \times 1000$, so the microvolts from the antenna become millivolts. The signal is now strong enough to not be degraded (in signal-to-noise ratio terms) during the further processing. The preamp usually contributes noise (sometimes most of the noise) but is the last component in the change to do so.

Filter: This reduces the range of frequencies present, defining a bandwidth and cutting out radio signals outside the band that might interfere with the observations.

Mixer: High frequency signals are difficult to amplify and manipulate. The mixer shifts the high radio frequency signals down to an intermediate frequency (IF) by mixing (i.e., multiplying) the signal, say $\mathrm{e}^{2\pi i \nu_0 t}$, with a local oscillator signal $\mathrm{e}^{-2\pi i \nu_{\mathrm{LO}} t}$ to give a resultant $\mathrm{e}^{2\pi i (\nu_0 - \nu_{\mathrm{LO}}) t}$.

Square-law detector We are interested in the power of the signal, not the voltage. Power is proportional to v^2, so we need a block that will carry out this squaring process. A diode (suitably inserted) can do this.

Integrator This averages the fluctuating noise power output of the detector to give a signal proportion to its mean level. This improves the signal-to-noise ratio.

[2 marks each. The order of the filter and mixer are not important].

(c) Why is it important that the resolution of the antenna is low enough for the radiogalaxy to lie entirely within the beam of the antenna? **[2]**

Solution: If the source is lager than the antenna beam then the effective flux of the source is reduced and the signal-to-noise ratio (and the chances of detection) is reduced. The telescope would be resolving the source and therefore only seeing part of it in any single observation.

(d) Two of these antennas are separated on a baseline $\boldsymbol{D}$, and used as an interferometer to observe the same source. Given that $P \simeq 1$ over the source, show that the correlated signal from these antennas is

$$\langle \psi_1 \psi_2^* \rangle \propto \int B(\boldsymbol{s}) \exp(i k \boldsymbol{D} \cdot \boldsymbol{s}) \, \mathrm{d}\Omega ,$$

where ψ_1 and ψ_2 are the signal voltages from each antenna and k is the wavenumber of the radiation received from the source. **[6]**

Solution: If this were a point source of flux density S, the two antennas would see exactly the same signal but with a delay length of $\boldsymbol{D} \cdot \boldsymbol{s}$ between them. If $\langle |\psi_1|^2 \rangle = \langle |\psi_2|^2 \rangle \propto S$ then

$$\langle \psi_1 \psi_2^* \rangle = S \exp(i k \boldsymbol{D} \cdot \boldsymbol{s}).$$

For an extended incoherent source the correlation between signals from two different patches of sky is zero. A small patch of sky of surface brightness $B(\boldsymbol{s})$ and solid angle $\mathrm{d}\Omega$

Q 4 continued

will contribute a flux density of $B(\boldsymbol{s})\,\mathrm{d}\Omega$, so we can write its differential contribution to the overall correlation as

$$\mathrm{d}\langle\psi_1\psi_2^*\rangle = B(\boldsymbol{\theta})\,\mathrm{d}\Omega\,\exp(ik\boldsymbol{D}\cdot\boldsymbol{s}).$$

Because the patches are incoherent, the total correlation from the entire sky is just the sum (or integral) of the correlations from all the patches, i.e.,

$$\langle\psi_1\psi_2^*\rangle \propto \int B(\boldsymbol{s})\exp(ik\boldsymbol{D}\cdot\boldsymbol{s})\,\mathrm{d}\Omega.$$

(e) The radiogalaxy looks like two diffuse blobs of emission, each of angular size α separated by an angle $\beta\,(>\alpha)$. Without detailed calculation, sketch how you would expect the amplitude of the correlated signal, $|\langle\psi_1\psi_2^*\rangle|$ (also known as the fringe visibility) to change as a function of D/λ, justifying the features you sketch. **[5]**

[Detailed calculations are not required, but you may like to consult the Fourier transform relation $\int_{-\infty}^{\infty}\mathrm{e}^{2\pi isu}\mathrm{e}^{-s^2/\alpha^2}\,\mathrm{d}s \propto \mathrm{e}^{-\pi^2\alpha^2u^2}$.]

Solution: The question does not demand an analytical solution, but we can model the blobs as gaussian, so that

$$B(s) \propto \exp\left[-\frac{(s+\beta/2)^2}{\alpha^2}\right] + \exp\left[-\frac{(s-\beta/2)^2}{\alpha^2}\right].$$

This is the convolution of a single gaussian and two δ-functions separated by β (i.e., at $\pm\beta/2$.). The fringe visibility (proportional to the correlated signal) is the Fourier transform of the (normalised) sky brightness distribution, so is the *product* of the transform of the gaussian and the transform of the two δ-functions.

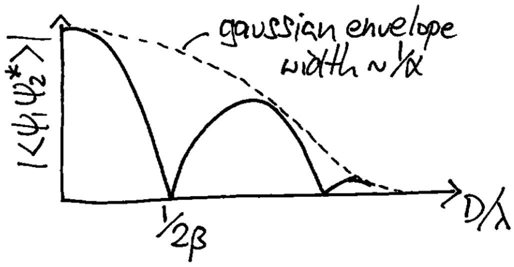
$$\mathrm{FT}[\delta(s-\beta/2)+\delta(s+\beta/2)] \propto \mathrm{e}^{ik\beta D/2}+\mathrm{e}^{-ik\beta D/2} \propto \cos(\pi\beta D/\lambda).$$

Similarly, using the hint,

$$\mathrm{FT}[\exp(-s^2/\alpha^2)] \propto \exp(-\pi^2\alpha^2D^2/\lambda^2),$$

So the resultant fringe visibility is

$$|\langle\psi_1\psi_2^*\rangle| \propto \Gamma(D) = \exp(-\pi^2D^2\alpha^2/\lambda^2)|\cos(\pi D\beta/\lambda)|.$$



Q 4 continued

The sketch should therefore look like a cosine in D/λ ($D > 0$), with its first zero at $D/\lambda = 1/(2\beta)$ and a gaussian envelope of width $\sim 1/\alpha$. Any sketch that shows the fringe zeros spaced in a way proportional to $1/\beta$ and an envelop of characteristic width $1/\alpha$ is acceptable.

[Total: 30]

5 (a) A single-dish radio telescope is used to monitor the flux density of a compact radio source over several hours. Define what is meant by the telescope's *system temperature*, describing how it arises and how it affects the sensitivity of the measurement. **[4]**

Solution: It is useful to measure broadband noise as an equivalent temperature, such that the power present per unit bandwidth has a value of $k_B T$. The noise in a radio telescopes comes from many sources. One is the signal itself, as celestial radio signals are generally broadband noise sources. In addition there are further contributions to the noise from galactic emission in the telescope beam, from ground spillover into the beam, from the atmosphere and from the electronics in the telescope system itself. The sum of all these contributions is called the system temperature, and it defines the overall uncertainty in the measurement of a flux density. However, if one observes for a time τ over a bandwidth of $\Delta\nu$, the standard deviation from this noise is reduced by a factor of $(\Delta\nu\tau)^{1/2}$ (corresponding to the square root of the number of independent samples that have been averaged together). Low system temperatures, wide bandwidths and long integration times therefore lead to sensitive measurements.

(b) Why is it useful to 'chop' on and off the source during these observations? **[2]**

Solution: The gain and system temperature of a radio telescope fluctuate in time, sometimes on very short timescales (less than a second). This makes it difficult to determine the contribution to the overall temperature from the signal itself. One solution to this is to rapidly point the telescope towards and away from the source, and attribute the difference as the contribution from the source. This is chopping, or beamswitching.

(c) An observation is carried out in which the flux density of the source is estimated by subtracting the 'off-source' signal from the 'on-source' signal. By considering the overall uncertainty in the difference signal, show that if one spends half the overall observing time on-source, the final sensitivity is one half what could be achieved if chopping were unnecessary and all the noise was from the source. **[4]**

Solution: Let the contribution to the overall system temperature form the source be T. In the situation where chopping is not necessary, and the only contribution to the noise if from the source itself (i.e., $T_{\text{on}} = T$), the uncertainly (standard deviation) in the measurement after a time τ_0 is

$$\text{SD}(T_{\text{on}}) = \frac{T_{\text{on}}}{\sqrt{\Delta\nu\tau_0}}.$$

When there are other contributions to the system temperature (and $T \ll T_{\text{sys}}$), the on-source and off-source temperatures are related by

$$T_{\text{on}} = T_{\text{off}} + T.$$

Q 5 continued

As each is observed for a time $\tau_0/2$, the standard deviations of their estimates are

$$\mathrm{SD}(T_{\mathrm{on}}) = T_{\mathrm{on}}(\Delta\nu\tau_0/2)^{-1/2},$$
$$\mathrm{SD}(T_{\mathrm{off}}) = T_{\mathrm{off}}(\Delta\nu\tau_0/2)^{-1/2} \simeq T_{\mathrm{on}}(\Delta\nu\tau_0/2)^{-1/2}.$$

So the variance of our estimate of T is

$$\mathrm{var}(T) = \mathrm{var}(T_{\mathrm{on}}) + \mathrm{var}(T_{\mathrm{off}}) \simeq \frac{4T_{\mathrm{on}}^2}{\Delta\nu\tau_0}.$$

The uncertainty in our estimate of T is therefore

$$\mathrm{SD}(T_{\mathrm{on}}) = 2\frac{T_{\mathrm{on}}}{\sqrt{\Delta\nu\tau_0}},$$

which is twice our previous result.

[Total: 10]

6

(a) The one-dimensional van Cittert-Zernike theorem relating the complex fringe visibility Γ, measured by a correlating radio interferometer, to the normalised sky brightness $B_{\rm n}$ over a small patch of sky can be written as

$$\Gamma(y) = \int B_{\rm n}(\alpha) \exp(iky\alpha)\, {\rm d}\alpha,$$

where y is the (projected) distance between the two telescopes in the interferometer, k is the wavenumber of the radiation and α is a small angle. Briefly describe the relevance of this expression in imaging radio astronomy and give two reasons why interferometry is usually the preferred method for imaging the sky. **[6]**

Solution: The VCZ theorem relates the complex fringe visibility, computed as the correlation between signals from two elements of an interferometer, to the sky brightness. It is a fourier transform, and implies that a map of the sky can be created by applying an inverse fourier transform to the measured visibilities. The relationship is central to imaging interferometry as a result. However, in practice only a fraction of the transform plane is sampled, and the expression also breaks down for large angles from the phase centre. [. . . 4]

Images in radio astronomy are usually created using interferometry. Interferometry offers better spatial resolution than single-dish 'raster' methods, giving an angular resolution of the order of λ/D where D is the baseline length. In addition, interferometry solves several problems to do with instrument stability. An interferometer is less susceptible to gain fluctuations in the amplifier chain, and there is no need to 'chop' on and off a source in an interferometer. [. . . 2]

(b) A radio source consists of two bright points separated by an angle α_0. Determine and sketch the modulus of the fringe visibility from this source as a function of antenna separation if

i. the two points are equally bright **[5]**

ii. one point is twice as bright as the other. **[5]**

How would you expect these results to change if the sources were resolvable disks rather than points? **[3]**

Solution: Without loss of generality, let the two sources have flux densities of r and 1 (so that r will later have the values of 1 and 2). Our normalised sky brightness therefore has the form

$$B_{\rm n}(\alpha) = \frac{1}{1+r}[\delta(\alpha - \alpha_0/2) + r\delta(\alpha + \alpha_0/2)].$$

Inserting into our expression for Γ we get

$$(1+r)\Gamma(y) = {\rm e}^{-iky\alpha_0/2} + r{\rm e}^{iky\alpha_0/2}.$$

Q 6 continued

Taking the modulus of both sides

$$(1+r)^2|\Gamma(y)|^2 = 1 + r^2 + 2\mathrm{Re}[re^{iky\alpha_0}]$$
$$\text{so} \quad |\Gamma(y)|^2 = \frac{1 + r^2 + 2r\cos ky\alpha_0}{(1+r)^2}.$$

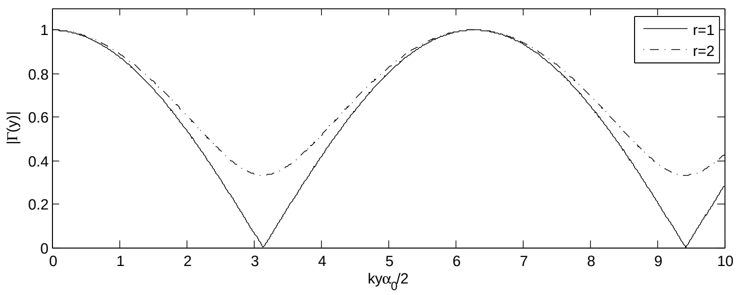
If $r = 1$ (case i) then

$$|\Gamma(y)|^2 = \frac{2(1+\cos ky\alpha_0)}{4}$$
$$= \cos^2 ky\alpha_0/2$$
$$\text{so} \quad |\Gamma(y)| = |\cos ky\alpha_0/2|.$$

This has a maximum value of 1 and its first zero at $y = \frac{\lambda}{2\alpha_0}$.

If $r = 2$ then

$$|\Gamma(y)|^2 = \frac{1 + 4 + 4\cos ky\alpha_0}{9}$$
$$\text{so} \quad |\Gamma(y)| = \frac{1}{3}(5 + 4\cos ky\alpha_0)^{1/2}.$$



If the sources were disks then the visibility would be the product of the above and the fourier transform of a disk, and so would drop away at large y values.

(c) LOFAR is a large low-frequency radio telescope interferometer consisting of many antenna stations around Europe. Each station is an array of 96 dipole antennas, closely packed within a circle. Distinguish between the idea of an antenna array and an interferometer. **[3]**

Solution: An antenna array is formed when the signals from several antennas are summed to make one large single antenna. The effective area of the sum is approximately the sum of the individual effective areas. In an interferometer, the signals from pairs of antennas are multiplied (rather than added), and the resulting correlation product used as a measure of a spatial frequency of the source.

Q 6 continued

(d) Given that the effective area of a single short antenna at a wavelength λ is

$$A_e = \frac{1}{3}\lambda^2,$$

show that the beamwidth of a single LOFAR station is approximately 10 degrees. **[4]**

Solution: The beam solid angle is

$$\Omega = \frac{\lambda^2}{NA_e}$$

where N is the number of antennas in the array. If $A_e = \frac{1}{3}\lambda^2$, then

$$\Omega = \frac{3}{N} = \frac{1}{32} \quad \text{steradians.}$$

As this is a small solid angle, the beamwidth is approximately the square root of the area (any result close to this is fine), which is 0.177 radians or 10 degrees

(e) The system temperature of a LOFAR station is dominated by galactic emission which shows a power law dependence on wavelength such that $T_{sky} \simeq 60\lambda^{2.5}$ kelvin, where λ is in metres. Show that the flux sensitivity of a station is therefore only weakly dependent on wavelength, and estimate its value in janskys for a 1-hour integration over a bandwidth of 6 MHz at 75 MHz. **[4]**

Solution: We use the equation of radio astronomy, giving the flux sensitivity of a telescope of area A and system temperature T as

$$S_{min} = \frac{2k_B T}{A\sqrt{\Delta\nu\tau}}.$$

Assuming $A = N\lambda^2/3 = 32\lambda^2$, we have

$$S_{min} = \frac{2k_B 60\lambda^{2.5}}{32\lambda^2\sqrt{\Delta\nu\tau}} = \frac{15k_B\lambda^{0.5}}{4\sqrt{\Delta\nu\tau}}.$$

If $\lambda = 4\,\text{m}$, $\tau = 3600\,\text{s}$ and $\Delta\nu = 6\,\text{MHz}$ then $S_{min} = 0.07\,\text{Jy}$.

[Total: 30]

End of Paper

NOTE: EXAM WITH SOLUTIONS