AA12M Statistical Astronomy (STA)
problem sheet #1

This problem sheet covers the introductory section of the course, including the fundamental ideas behind probability and the manipulation of logical and inferential statements. The second half of the sheet covers probability distribution functions (pdfs) and their manipulation, particularly variable transforms. Full solutions for all these problems will appear as the course progresses. Answers to some problems are shown in curly brackets.

1) Use the rules of Boolean algebra to simplify the following expressions:

(a) $AC + ABC$ $\{AC\}$

(b) $(A + B)(\overline{A} + B)$ $\{B\}$

(c) $A(\overline{A} + C) + C$ $\{C\}$

2) Express the following statements as Boolean expressions:

(a) C is true only if both A and B are true or if both A and B are false.

(b) D is true if any of A, B, and C are true, otherwise it's false.

3) Using the extended sum rule

$$P(A + B|C) = P(A|C) + P(B|C) - P(AB|C),$$

prove that for three propositions A_1, A_2 and A_3

$$\begin{aligned} P(A_1 + A_2 + A_3|C) = {} & P(A_1|C) + P(A_2|C) + P(A_3|C) \\ & - P(A_1A_2|C) - P(A_2A_3|C) - P(A_3A_1|C) \\ & + P(A_1A_2A_3|C). \end{aligned}$$

Hence prove that for n exhaustive, mutually exclusive propositions (i.e., a set of propositions for which one, and only one, is true),

$$P(A_1 + ... + A_n|B) = \sum_{i=1}^{n} P(A_i|B) = 1.$$

4) Using the result

$$\sum_{i=1}^{n} P(A_i|BC) = 1,$$

for exhaustive mutually exclusive propositions $\{A_i\}$, prove that

$$P(B|C) = \sum_{i=1}^{n} P(B|A_iC)P(A_i|C),$$

and, using the product rule, that

$$P(B|C) = \sum_{i=1}^{n} P(A_i B|C).$$

Hence prove that Bayes' Theorem can we written as

$$\begin{aligned} P(X|BC) &= \frac{P(X|C)P(B|XC)}{P(B|C)} \\ &= \frac{P(X|C)P(B|XC)}{\sum_i P(B|A_i C)P(A_i|C)} \\ &= \frac{P(X|C)P(B|XC)}{\sum_i P(BA_i|C)}. \end{aligned}$$

5) A bowl contains 8 coloured chips: 5 red and 3 green. A chip is drawn at random from the bowl, its colour noted, and then replaced in the bowl. This procedure is repeated a second time.

(a) What is the probability that both chips are the same colour? {17/32}

(b) What is the probability that the two chips are different colours? {15/32}

A single blue chip is then added to the bowl. Three chips are drawn in turn at random from bowl, but are *not* replaced before the next chip is drawn.

(c) What is the probability that the first chip is red and the second chip is green? {5/24}

(d) What is the probability that the first chip is blue, the second chip red and the third chip green? {5/168}

(e) What is the probability that at least one of the three chips is green? {16/21}

6) During a meteor shower, meteors fall at a rate of 15.7 per hour. What is the probability of observing less than 5 in a given period of 30 minutes? {0.109}

7)e (a) Explain the circumstances in which the Poisson distribution,

$$P(n|\lambda) = \exp(-\lambda)\lambda^n/n!\,,$$

correctly describes the probability of n events occurring, and identify the meaning of the λ term in the above equation. [3]

(b) There is a constant probability that a fire will occur at any time in Glasgow, and there are two fires per day on average. Write down the probability, $P(n)$, of n fires occurring on any one day. Each fire requires the presence of an engine for a day. What is the minimum number of fire engines required in Glasgow so that the probability that all fires on a given day are attended to is better than 90 percent? [7]

8) The probability distribution of the luminosities of the brightest cluster galaxies can be modelled as exponential, i.e.,

$$p(L) = A \exp\left[-\frac{(L - L_*)}{\Delta}\right], \quad L > L_*, \quad \Delta > 0,$$

where A, L_* and Δ are constant parameters.

(a) Determine the value of A which makes $p(L)$ a properly normalised pdf, i.e. one which integrates to unity.

(b) Using this pdf, determine the mean and variance of L. Hint: use integration by parts, or use the definition of the gamma function,

$$\Gamma(n) = \int_0^\infty x^{n-1} \mathrm{e}^{-x} \, \mathrm{d}x = (n-1)!$$

(c) Derive an expression for the cdf of L, $C(L)$.

(d) Using the cdf, calculate the median value of L. Is the median greater than, equal to or less than the mean?

9) The random variables X and Y have a joint pdf given by

$$p(x, y) = \mathrm{e}^{-y}, \quad 0 < x < y < \infty, \quad \text{zero elsewhere.}$$

(a) Determine the marginal pdf of X and Y.

(b) Determine the conditional pdf of X given Y and Y given X.

(c) Are X and Y independent random variables?

10)e (a) Explain what is meant by a *probability density function* (pdf) and explain briefly how it used to determine actual probabilities. [3]

(b) Given a pdf for x, $p(x)$, show that the pdf for $y = f(x)$ is

$$p(y) = p(x) \left| \frac{\mathrm{d}x}{\mathrm{d}f} \right|$$

where x maps uniquely onto y. [4]

(c) Pulsars travel at a constant speed, u, and the pdf for the speed of any particular pulsar is $p(u) = \exp(-u)$. Calculate the expected distance a pulsar (of unknown speed) will travel in a time T. [5]

(d) What do we mean when we talk about the probabilities of *dependent* and *independent* quantities? How can we test whether or not two quantities are independent? [5]

(e) Two pulsars are seen in the same area of the sky. What is the probability distribution for the speed of the *slower* pulsar? (Hint: look at the previous problem...) [8]

(f) What is the probability distribution for the speed of the *faster* pulsar? [5]

11) The random variables X_1 and X_2 have joint pdf of

$$p(x_1, x_2) = 12x_1x_2(1 - x_2), \quad 0 < x_1 < 1, \quad 0 < x_2 < 1, \quad \text{zero elsewhere.}$$

Show that X_1 and X_2 are statistically independent.

12) The random variables X_1 and X_2 have joint pdf of

$$p(x_1, x_2) = 2\mathrm{e}^{-x_1 - x_2}, \quad 0 < x_1 < x_2, \quad 0 < x_2 < \infty, \quad \text{zero elsewhere.}$$

Show that X_1 and X_2 are statistically dependent.

13) Let X have the pdf

$$p(x) = x^2/9, \quad 0 < x < 3, \quad \text{zero elsewhere.}$$

Find the pdf of $Y = X^3$.

14) Let X have the pdf

$$p(x) = 2x\mathrm{e}^{-x^2}, \quad 0 < x < \infty, \quad \text{zero elsewhere.}$$

Find the pdf of $Y = X^2$.

15) Let X have a uniform pdf over the interval $(-\pi/2, \pi/2)$.

(a) Show that $Y = \tan X$ has the pdf (known as the *Cauchy* or *Lorentzian* distribution) of

$$p(y) = \frac{1}{\pi(1 + y^2)}, \quad -\infty < y < \infty.$$

(b) Determine the mean and variance of Y.

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