The Poisson Distribution

We will take as our 'Poisson process' the arrival of photons at a detector, with a mean arrival rate μ. We expect, on average, $\mu\tau$ photons to be detected during an observing period τ. Photons will arrive at random times during this period, but we can imagine dividing τ into a very large number of sub-intervals (M of them) which are sufficiently short that they either contain one photon or no photons. The probability, p_1, that any one sub-interval contains a photon is the expected number of photons divided by the number of sub-intervals, i.e.,

$$p_1 = \frac{\mu\tau}{M}.$$

The probability, p_0, that the sub-interval does not contain a photon is simply

$$p_0 = 1 - p_1,$$

because one or other outcome is certain[1] ($p_0 + p_1 = 1$).

The probability of, say, the first N sub-intervals containing a photon and the remaining $M-N$ being empty is $p_1^N p_0^{M-N}$. This is just one way in which N photons can be distributed in the interval. There are many other ways. In fact there are M ways to insert the first photon, $M-1$ ways to insert the second (avoiding the first), $M-2$ ways to insert the third and so on. There are therefore $M!/(M-N)!$ ways to insert all the photons. But we have over-counted. We have included identical configurations that have been arrived at merely by inserting photons in different orders. There are $N!$ ways to do this (the number of rearrangements of N objects), so the final number of arrangements of N photons in M sub-intervals is

$${}_M C_N = \frac{M!}{(M-N)!\,N!}.$$

Each of these configurations has an equal probability of happening ($= p_1^N p_0^{M-N}$), so the overall *probability* of receiving N photons is

$$p(N) = \frac{M!}{(M-N)!\,N!} p_1^N p_0^{M-N}.$$

This is known as the *binomial distribution*, and is valid for all values of M. However, we have assumed that M is very large, so we can consider its limiting form as $M \to \infty$ (and therefore as $p_1 \to 0$ with $Mp_1 = \mu\tau$). Without approximation, we can rewrite the above equation as

$$p(N) = \frac{(\mu\tau)^N}{N!} p_0^{M-N} f(M),$$

where

$$f(M) = \frac{M(M-1)(M-2)\dots(M-N+1)}{M^N}.$$

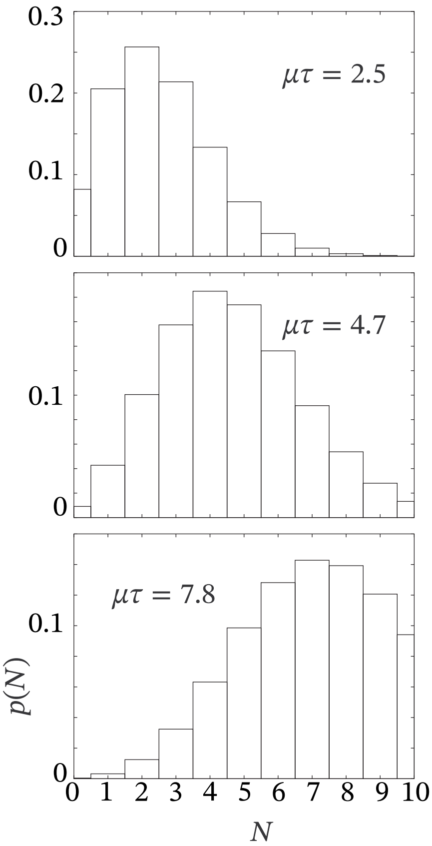

Figure 1: Poisson distributions for three mean arrival rates.

As $M \to \infty$, $f(M)$ approaches unity, as N becomes trivially small compared with M. We can also say that

$$\begin{aligned} p_0^{M-N} &= (1-p_1)^{M-N} \\ &= \left(1 - \frac{\mu\tau}{M}\right)^{M-N} \\ &= \frac{(1-\mu\tau/M)^M}{(1-\mu\tau/M)^N}. \end{aligned}$$

As $M \to \infty$ the numerator in the above expression approaches $\exp(-\mu\tau)$ and the denominator approaches unity. The final expression for $p(N)$ in the limit we require of large M is therefore

$$\boxed{p(N) = \exp(-\mu\tau)\frac{(\mu\tau)^N}{N!}}$$

which is our expression for the Poisson distribution.

GW
2021
v.584

[1]We have assumed that M is so big that the chances of two photons arriving in the same sub-interval is zero.