

The Poisson Distribution

We will take as our ‘Poisson process’ the arrival of photons at a detector, with a mean arrival rate μ . We expect, on average, $\mu\tau$ photons to be detected during an observing period τ . Photons will arrive at random times during this period, but we can imagine dividing τ into a very large number of sub-intervals (M of them) which are sufficiently short that they either contain one photon or no photons. The probability, p_1 , that any one sub-interval contains a photon is the expected number of photons divided by the number of sub-intervals, i.e.,

$$p_1 = \frac{\mu\tau}{M}.$$

The probability, p_0 , that the sub-interval does not contain a photon is simply

$$p_0 = 1 - p_1,$$

because one or other outcome is certain¹ ($p_0 + p_1 = 1$).

The probability of, say, the first N sub-intervals containing a photon and the remaining $M - N$ being empty is $p_1^N p_0^{M-N}$. This is just one way in which N photons can be distributed in the interval. There are many other ways. In fact there are M ways to insert the first photon, $M - 1$ ways to insert the second (avoiding the first), $M - 2$ ways to insert the third and so on. There are therefore $M!/(M - N)!$ ways to insert all the photons. But we have over-counted. We have included identical configurations that have been arrived at merely by inserting photons in different orders. There are $N!$ ways to do this (the number of rearrangements of N objects), so the final number of arrangements of N photons in M sub-intervals is

$${}_M C_N = \frac{M!}{(M - N)! N!}.$$

Each of these configurations has an equal probability of happening ($= p_1^N p_0^{M-N}$), so the overall probability of receiving N photons is

$$p(N) = \frac{M!}{(M - N)! N!} p_1^N p_0^{M-N}.$$

This is known as the *binomial distribution*, and is valid for all values of M . However, we have assumed that M is very large, so we can consider its limiting form as $M \rightarrow \infty$ (and therefore as $p_1 \rightarrow 0$ with $Mp_1 = \mu\tau$). Without approximation, we can rewrite the above equation as

$$p(N) = \frac{(\mu\tau)^N}{N!} p_0^{M-N} f(M),$$

where

$$f(M) = \frac{M(M - 1)(M - 2) \dots (M - N + 1)}{M^N}.$$

As $M \rightarrow \infty$, $f(M)$ approaches unity, as N becomes triv-

¹We have assumed that M is so big that the chances of two photons arriving in the same sub-interval is zero.

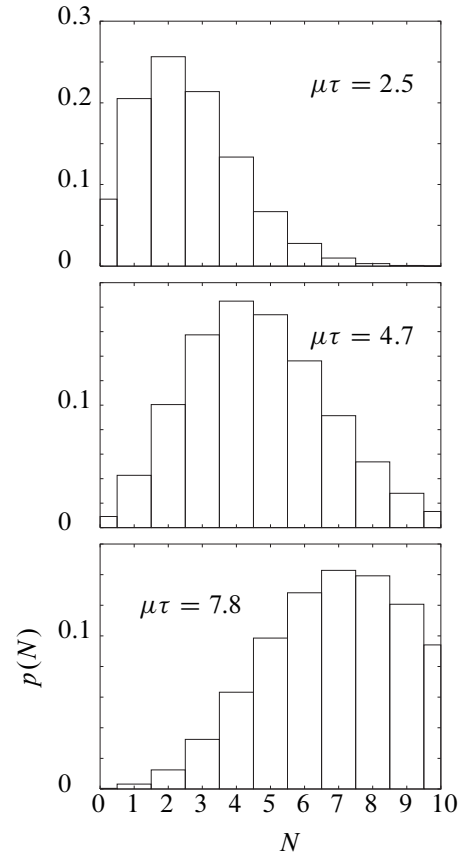


Figure 1: Poisson distributions for three mean arrival rates.

ially small compared with M . We can also say that

$$\begin{aligned} p_0^{M-N} &= (1 - p_1)^{M-N} \\ &= \left(1 - \frac{\mu\tau}{M}\right)^{M-N} \\ &= \frac{(1 - \mu\tau/M)^M}{(1 - \mu\tau/M)^N}. \end{aligned}$$

As $M \rightarrow \infty$ the numerator in the above expression approaches $\exp(-\mu\tau)$ and the denominator approaches unity. The final expression for $p(N)$ in the limit we require of large M is therefore

$$p(N) = \exp(-\mu\tau) \frac{(\mu\tau)^N}{N!}$$

which is our expression for the Poisson distribution.