The Maxwell-Boltzmann distribution – some useful background maths

If a classical* gas of freely-moving particles, each of mass m, is in thermal equilibrium at a temperature T, the speeds of the particles will settle into a *Maxwell-Boltzmann distribution*. If $f(v)\,\delta v$ is the fraction of particles with speeds in the range v to $v + \delta v$ then this distribution function is

$$f(v) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{kT}\right)^{3/2} v^2 \exp\left(-mv^2/2kT\right).$$

To derive this equation, and to work out the mean speed, $\langle v \rangle$, and the mean squared speed, $\langle v^2 \rangle$, of the particles we need to evaluate integrals of the form

$$I_n = \int_0^\infty x^n e^{-ax^2}\, dx.$$

We can tackle this first by integrating by parts:

$$\begin{aligned} I_n &= -\frac{1}{2a} \int_0^\infty x^{n-1}(-2ax) e^{-ax^2}\, dx \\ &= -\frac{1}{2a} \left[x^{n-1} e^{-ax^2}\right]_0^\infty \\ &\qquad + \frac{1}{2a} \int_0^\infty (n-1) x^{n-2} e^{-ax^2}\, dx \\ &= \frac{n-1}{2a} I_{n-2}. \end{aligned}$$

This is known as a *reduction formula*, and using it we can (eventually) evaluate any integral I_n we need if we know I_0 and I_1.

I_1 can be integrated directly:

$$\begin{aligned} I_1 &= \int_0^\infty x e^{-ax^2}\, dx \\ &= -\frac{1}{2a} \left[e^{-ax^2}\right]_0^\infty \\ &= \frac{1}{2a}. \end{aligned}$$

*The gas is classical if quantum effects can be neglected. Here that means that the box containing the gas is much bigger than any particle's de Broglie wavelength.

I_0 is a little more tricky. To evaluate it, first consider the square of the integral, written as

$$\begin{aligned} I_0^2 &= \int_0^\infty e^{-ax^2}\, dx \times \int_0^\infty e^{-ay^2}\, dy \\ &= \iint_0^\infty e^{-a(x^2+y^2)}\, dx\, dy. \end{aligned}$$

This is the volume under a two-dimensional gaussian curve, over the positive quarter of the (x, y) plane. In polar coordinates we can write this same volume in terms of r, where $r^2 = x^2 + y^2$, as

$$\begin{aligned} I_0^2 &= \frac{1}{4} \int_0^\infty 2\pi r e^{-ar^2}\, dr \\ &= -\frac{\pi}{4a} \left[e^{-ar^2}\right]_0^\infty \\ &= \frac{\pi}{4a}. \end{aligned}$$

So finally, we can present the full reduction formula:

If

$$I_n = \int_0^\infty x^n e^{-ax^2}\, dx$$

then

$$\begin{aligned} I_0 &= \frac{1}{2}\sqrt{\frac{\pi}{a}} \\ I_1 &= \frac{1}{2a} \\ I_n &= \frac{n-1}{2a} I_{n-2} \end{aligned}$$

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