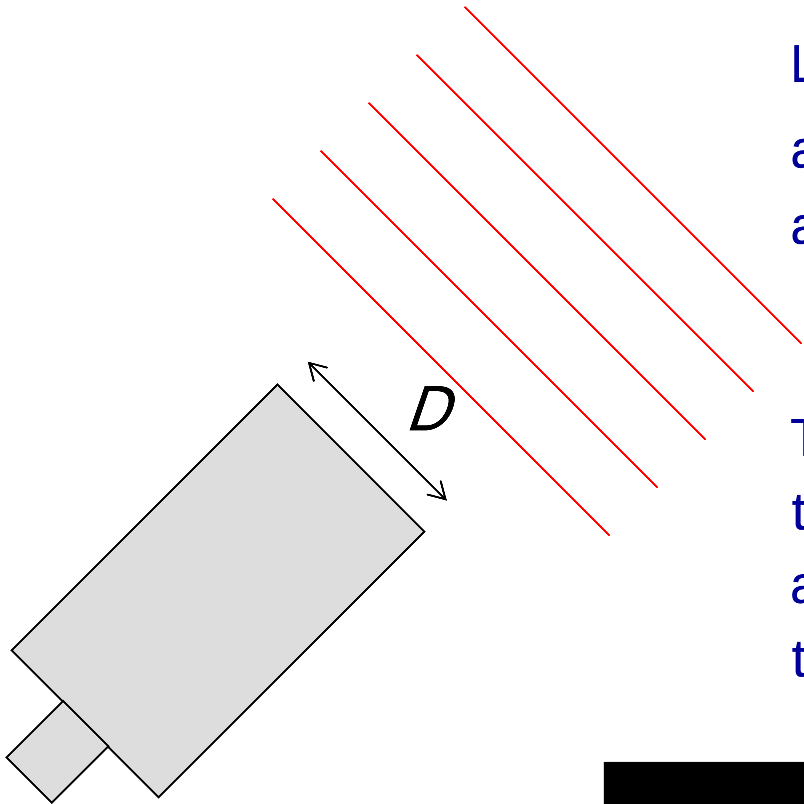
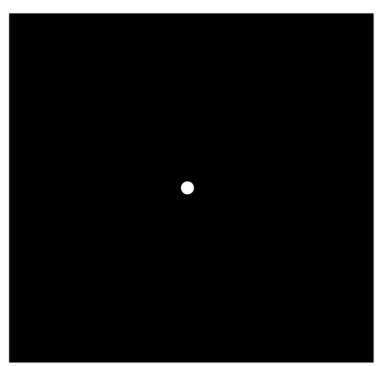
7. Resolving Power and Interferometry

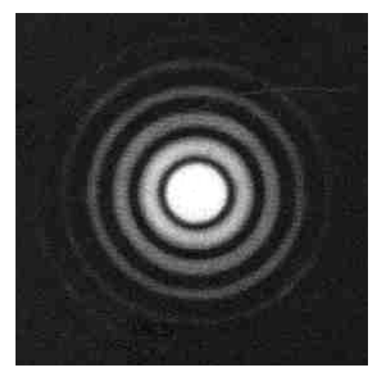


Light from a distant point source arrives at a telescope aperture (of diameter *D*) as plane waves.

These are diffracted as they pass through the circular aperture to give the point source an apparent ‘shape’ first analysed theoretically by Airy.



point source

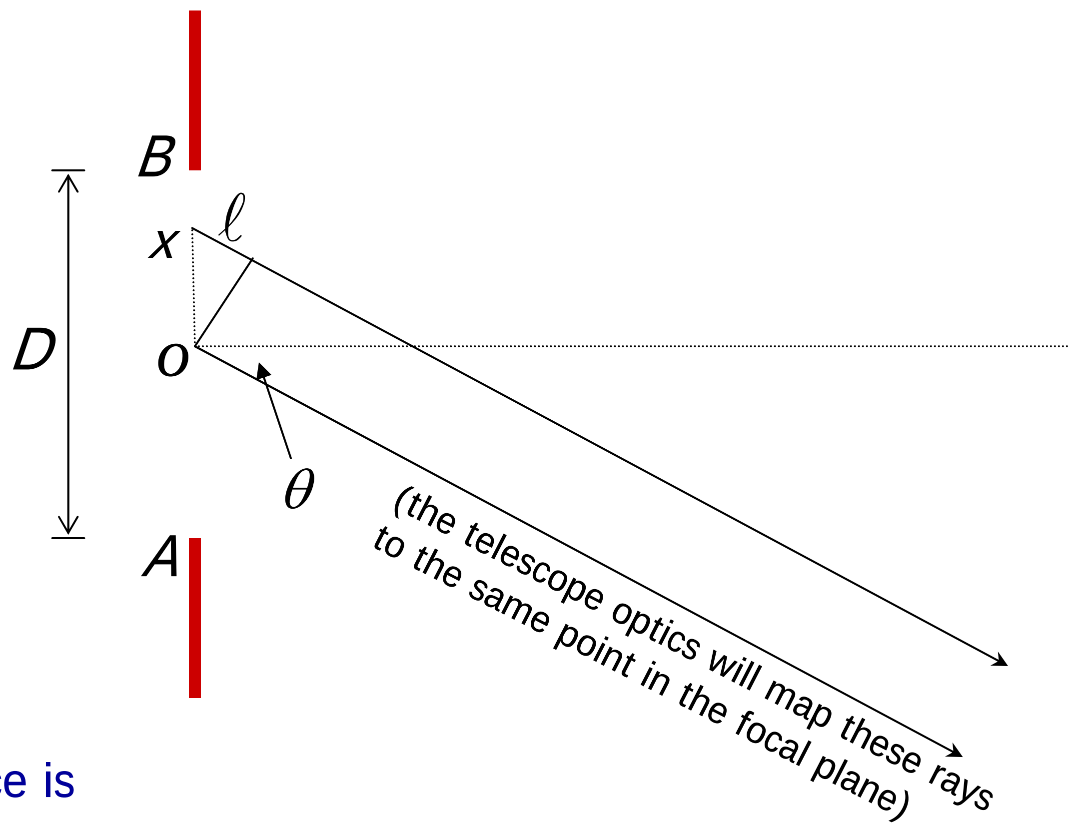


Airy disk

We can work out how the width of the Airy disk pattern depends on the size of the telescope aperture and the wavelength of the incident light.

Simplified 1-d analysis:

Integrate diffraction pattern across a 'slit' of width D



Consider light emerging from the aperture at angle θ to the axis:

Path difference between light from o and x is $\ell = x \sin\theta$

The corresponding phase difference is

$$\phi(x) = \frac{2\pi}{\lambda} x \sin\theta \quad \approx \quad \frac{2\pi\,\theta}{\lambda} x \quad \text{for small } \theta \qquad (7.1)$$

Compare to the wave emerging from O the wave from X has a complex amplitude of

$$\psi(x) = e^{i\phi} = e^{i\frac{2\pi\theta}{\lambda}x} \tag{7.2}$$

By the principle of superposition, the total diffracted signal at an angle θ is obtained by integrating eq. (7.2) from $x = -D/2$ to $x = D/2$

$$\boxed{\psi_{\text{tot}}(\theta) = \int_{-D/2}^{D/2} e^{i\frac{2\pi\theta}{\lambda}x}\, dx} \tag{7.3}$$

Integrating gives

$$\psi_{\text{tot}}(\theta) = \frac{\lambda}{2\pi i\theta}\left[e^{i\pi\theta D/\lambda} - e^{-i\pi\theta D/\lambda}\right] \qquad (7.4)$$

which we can rewrite as

$$\psi_{\text{tot}}(\theta) = \frac{\lambda}{\pi\theta}\frac{1}{2i}\left[e^{i\pi\theta D/\lambda} - e^{-i\pi\theta D/\lambda}\right] \qquad (7.5)$$

or as

$$\psi_{\text{tot}}(\theta) = \frac{\lambda}{\pi\theta}\sin\left(\frac{\pi\theta D}{\lambda}\right) \qquad (7.6)$$

This in turn can be rewritten as

$$\psi_{\text{tot}}(\theta) = D\,\frac{\sin\left(\frac{\pi\theta D}{\lambda}\right)}{\frac{\pi\theta D}{\lambda}} \qquad (7.7)$$

and the intensity:

$$I(\theta) = \psi_{\text{tot}}\,\psi^{*}_{\text{tot}} \quad \Rightarrow \quad \boxed{I(\theta) = D^2\,\frac{\sin^2\left(\frac{\pi\theta D}{\lambda}\right)}{\left(\frac{\pi\theta D}{\lambda}\right)^2}} \qquad (7.8)$$

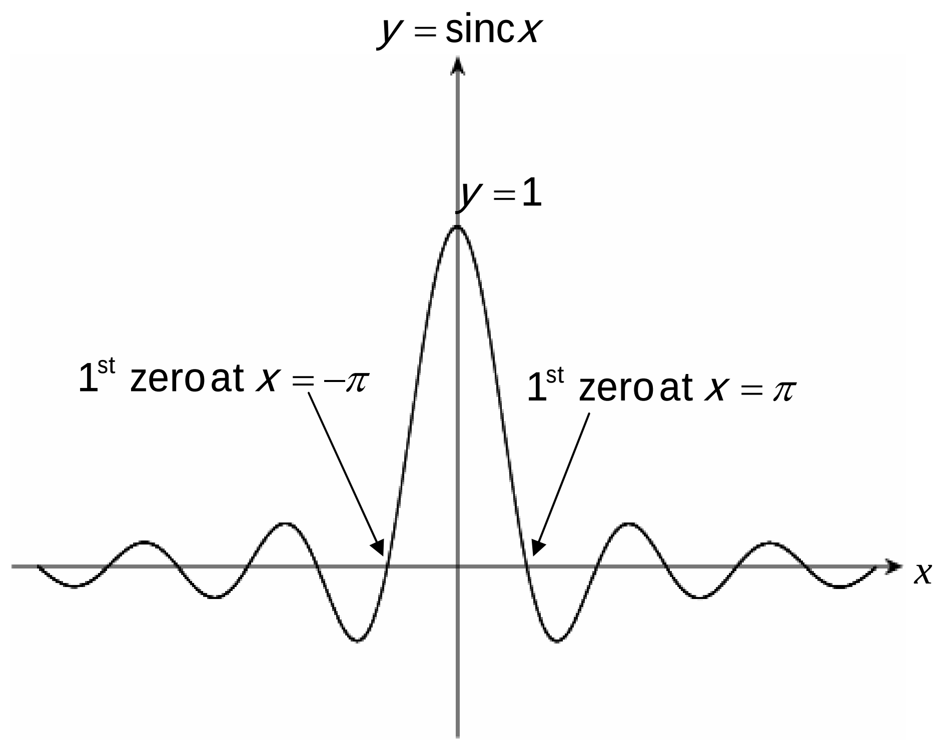
We can write eq. 7.8 as $$\boxed{I(\theta) = I_0 \operatorname{sinc}^2\left(\frac{\pi \theta D}{\lambda}\right)} \qquad (7.9)$$

Where I_0 is the intensity at $\theta = 0$ and $\operatorname{sinc} x \equiv \dfrac{\sin x}{x}$

[sinc x can also be defined as (sin πx)/(πx)]

The sinc function occurs frequently in optics

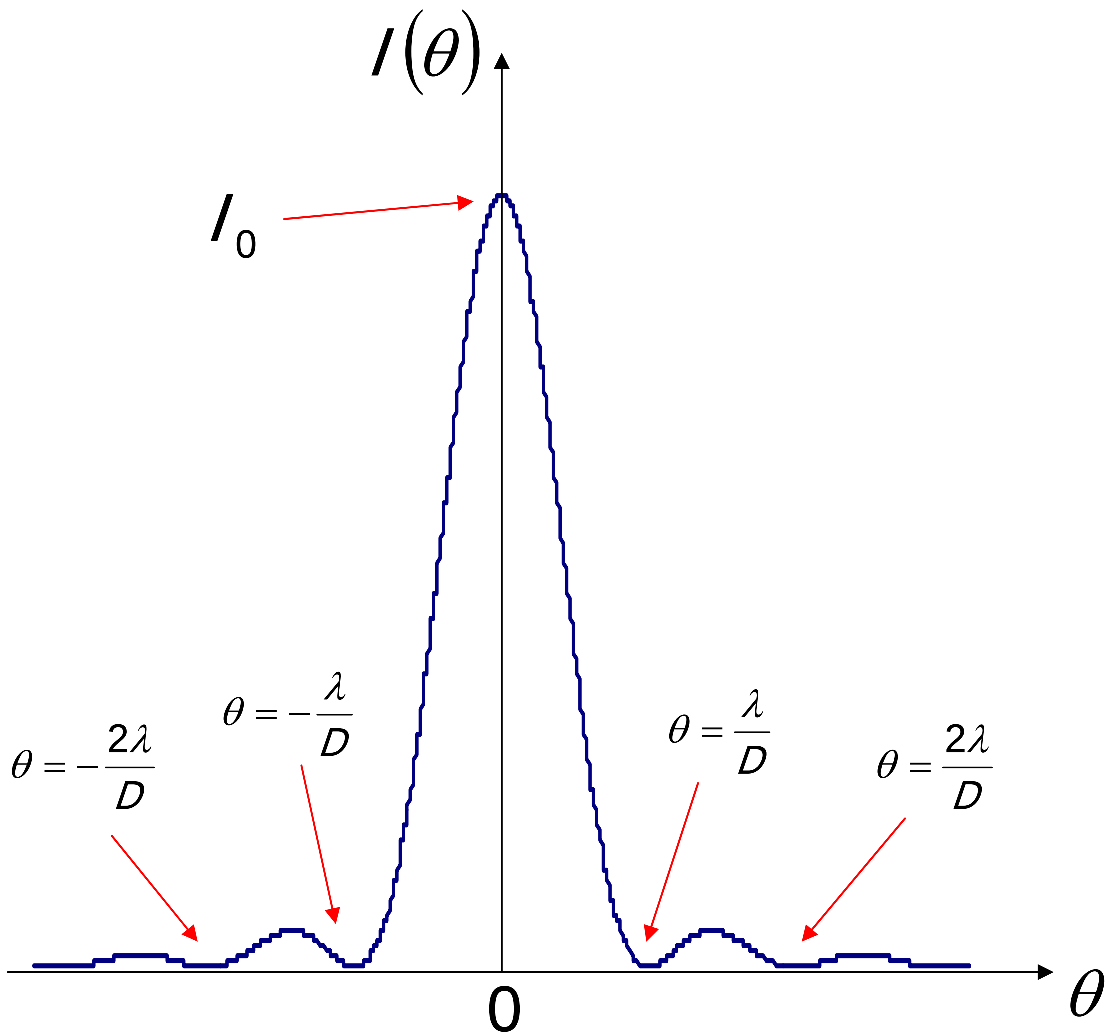
The function has a maximum at $x = 0$ and the zeros occur at $x = \pm m\pi$ for positive integer m



For our simplified 1-d analysis:

Intensity pattern has minima at $\frac{\pi\theta D}{\lambda} = \pm m\pi$ i.e. at

$$\boxed{\theta = \pm m\frac{\lambda}{D}} \quad (7.10)$$



More exact 2-d analysis:

Integrate diffraction pattern over a circular aperture of diameter D

$$I(\theta) = I_0 \left(\frac{2J_1(ka\sin\theta)}{ka\sin\theta} \right)^2 \qquad \text{where} \qquad k = \frac{2\pi}{\lambda}$$

$J_1(x)$ is a Bessel function of the first kind, a is the radius of the disc. The first zero of $J_1(x)$ is where x = 3.83, so the first zero of the diffraction pattern is where

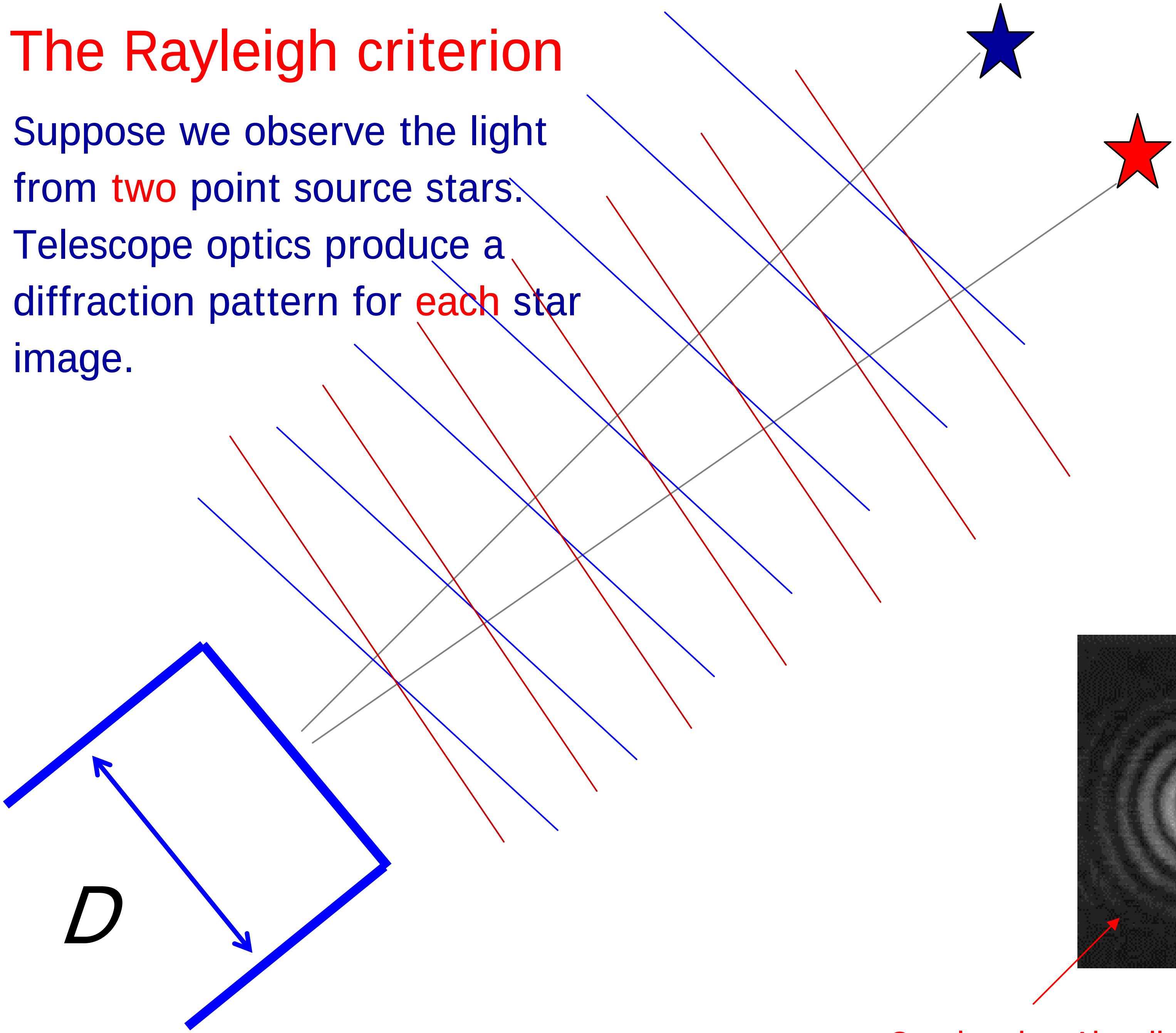
$$\sin\theta = \frac{3.83}{ka} = \frac{3.83\lambda}{\pi d} = 1.22\frac{\lambda}{d}$$

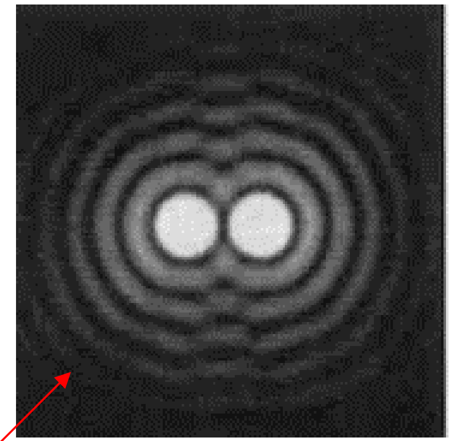
First minimum at

$$\boxed{\theta \approx \pm\frac{1.22\,\lambda}{D}} \qquad (7.11)$$

The Rayleigh criterion

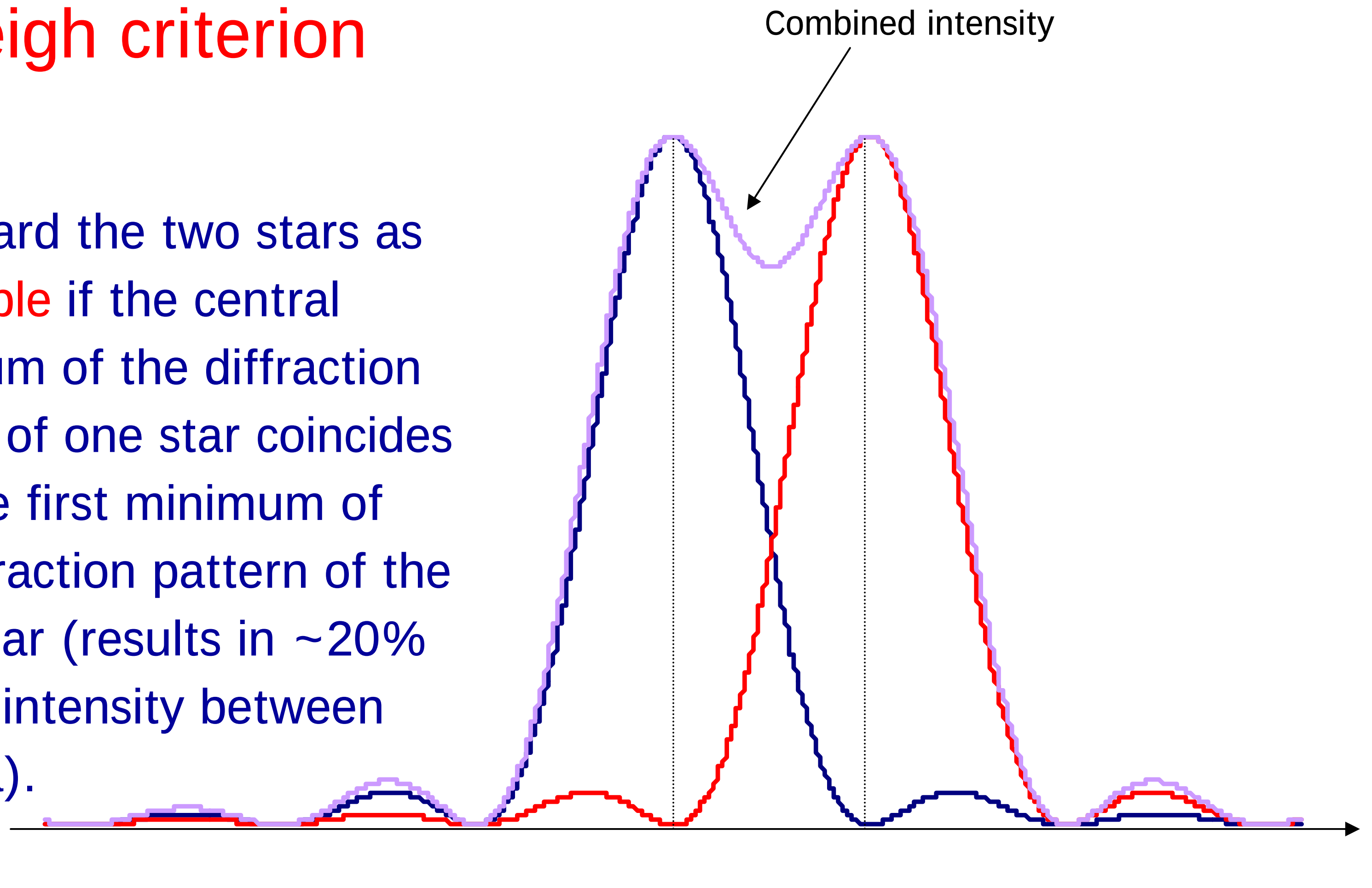
Suppose we observe the light from two point source stars. Telescope optics produce a diffraction pattern for each star image.





Overlapping Airy disks

The Rayleigh criterion



We regard the two stars as resolvable if the central maximum of the diffraction pattern of one star coincides with the first minimum of the diffraction pattern of the other star (results in ~20% drop in intensity between maxima).

From eq. (7.11), the two stars are resolvable if their angular separation satisfies:

$$\boxed{\theta_{\text{min}} = \frac{1.22\,\lambda}{D}} \qquad (7.12)$$

in radians

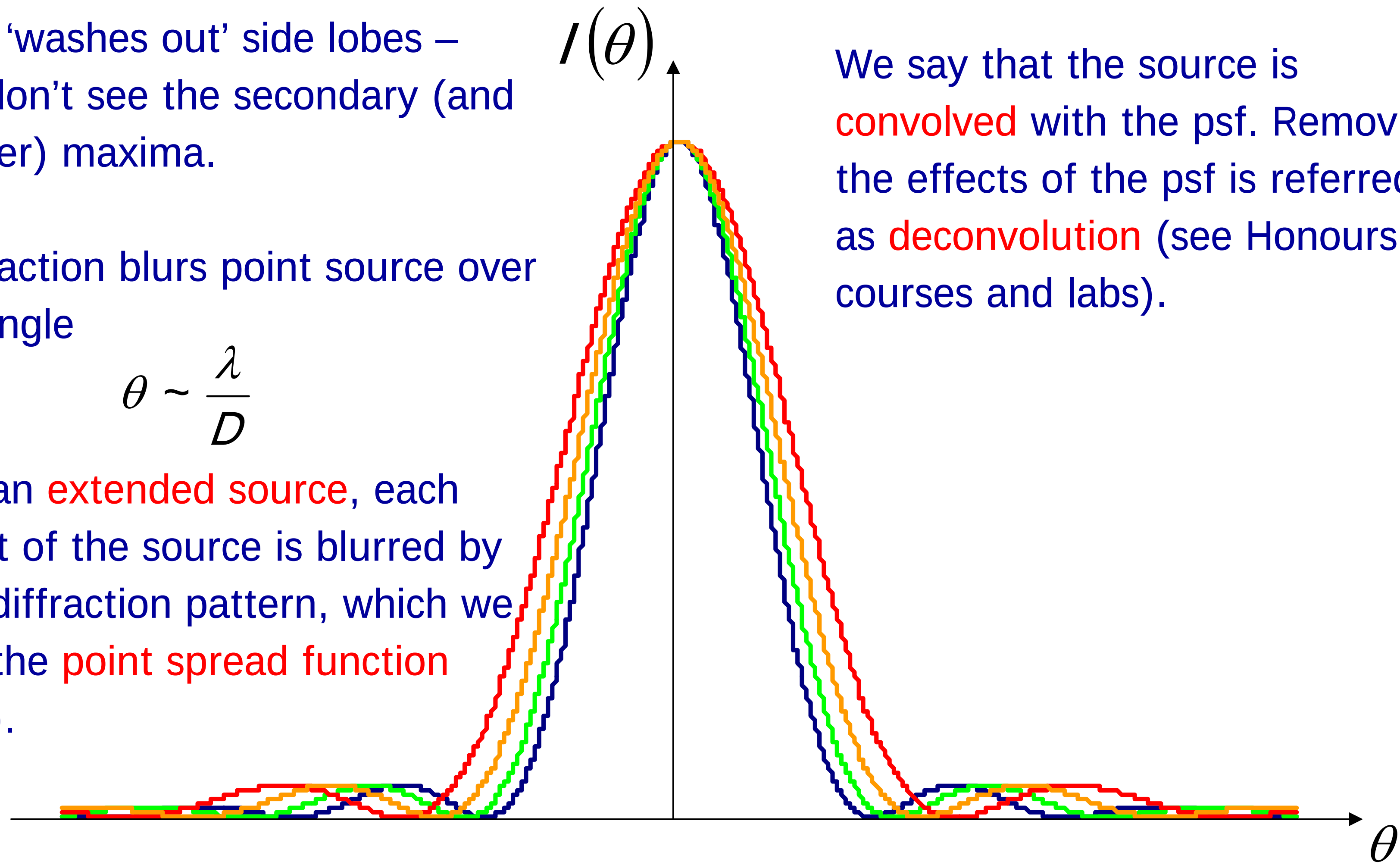
If we observe a point source which is not monochromatic (e.g. any unfiltered star), the observed intensity is the sum (integral) of the intensity pattern at each observed wavelength:

This 'washes out' side lobes – we don't see the secondary (and higher) maxima.

Diffraction blurs point source over an angle

$$\theta \sim \frac{\lambda}{D}$$

For an extended source, each point of the source is blurred by the diffraction pattern, which we call the point spread function (psf).



We say that the source is convolved with the psf. Removing the effects of the psf is referred to as deconvolution (see Honours courses and labs).

Equation (7.12) defines the theoretical angular resolving power of a telescope.

If we can resolve features down to $\theta_{\min}$ we say that the telescope is diffraction limited.

e.g., for $\lambda = 550$ nm , $\theta_{\min} = \dfrac{0.14}{D_{\text{in metres}}}$ arcsec (7.13)

For small (amateur) telescopes, of aperture a few cm, $\theta_{\min}$ is larger than the typical size of the seeing disk (Section 5), so these telescopes are not limited by seeing.

For ground-based optical telescopes with $D \geq 1\ \mathrm{m}$, on the other hand, we find that $\theta_{\min}$ is much *smaller* than the seeing disk. Hence, the theoretical angular resolving limit is never achieved, and we say that the telescope is seeing limited.

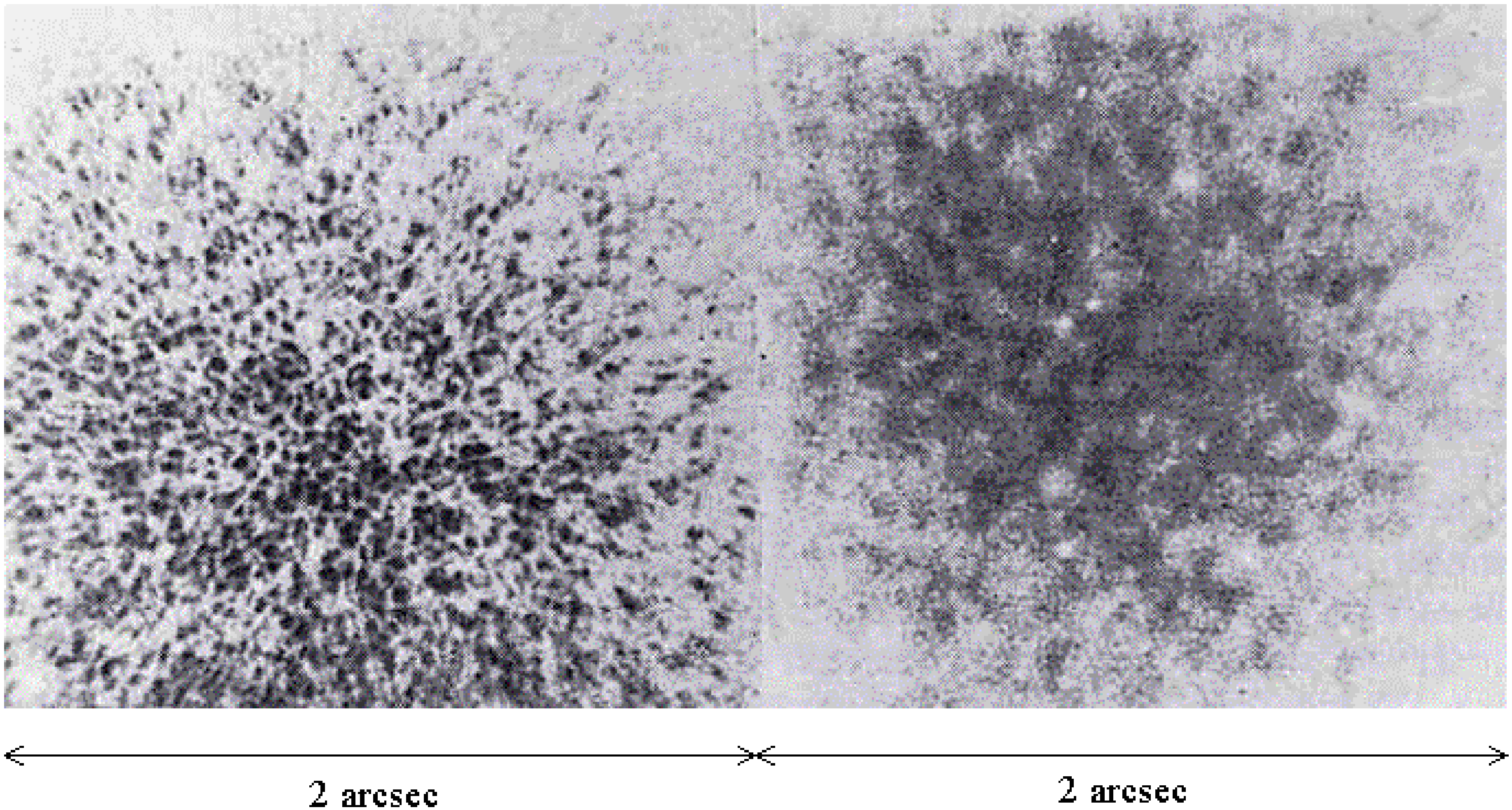
Speckle interferometery

- We saw in Section 5 that turbulence in the atmosphere causes scintillation, which smears out light into seeing disk over time.

- With a large telescope and a sensitive detector we can collect enough photons from a bright (i.e. $m_V < 10$) source to get a good SNR from an exposure of only a few milliseconds.

- Such a short exposure time ‘freezes’ the atmospheric scintillation: our image is still affected by a pattern of hundreds of more/less dense cells of air – but these are not moving. We get a ‘snapshot’ of hundreds of light and dark spots caused by the interference between light shining through these cells -- a SPECKLE PATTERN over an area roughly equal to the seeing disk.

- Each speckle is a diffraction-limited image of the source, and Fourier analysis of their pattern allows reconstruction of some of the original source intensity.



Speckle pattern from 0.02s exposure of point source star

Speckle pattern from 0.02s exposure of Betelgeuse (ang. diam. ~ 0.05 arcsec)