4. Sensitivity and noise

We define:

the sensitivity of an instrument or detector as the smallest signal that it can measure which is clearly not 'noise'.

(noise = random signal from some other source). We can measure the reliability of an observation via its signal-to-noise ratio (SNR):

$$\boxed{\mathrm{SNR} = \frac{\text{expected signal level}}{\text{expected noise level}}}$$

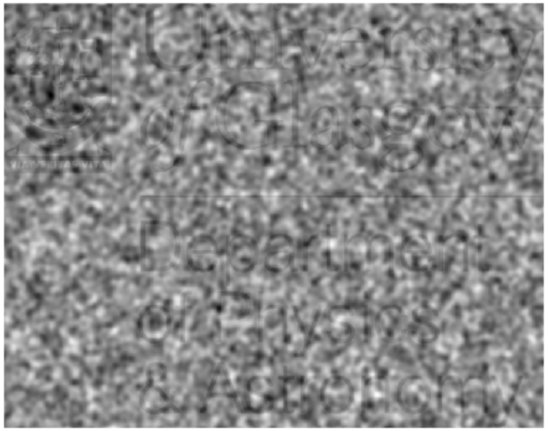
Generally, we don't trust observations unless the SNR is at least 3 (or preferably much greater).

Sometimes the signal might still be very weak, compared to a (removable) background.

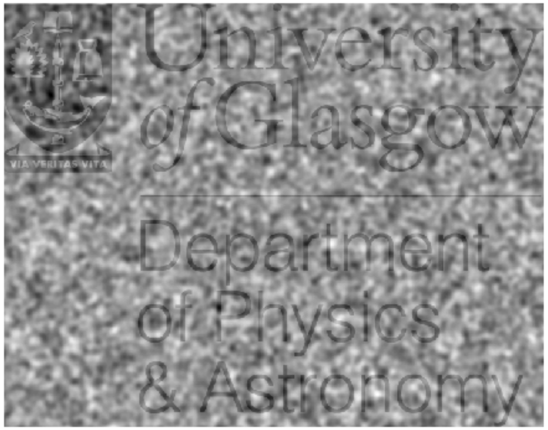
The effect of signal-to-noise ratio

SNR = 0.5

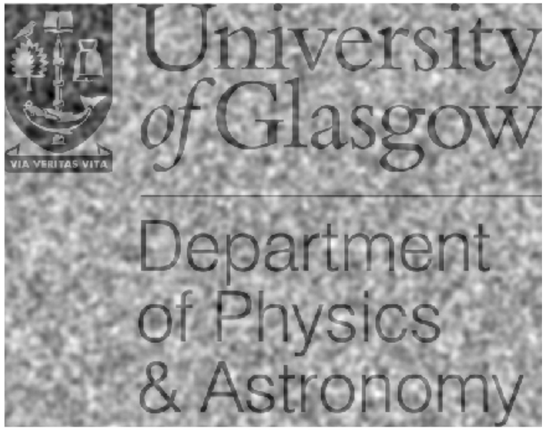
SNR = 1



SNR = 2



SNR = 4



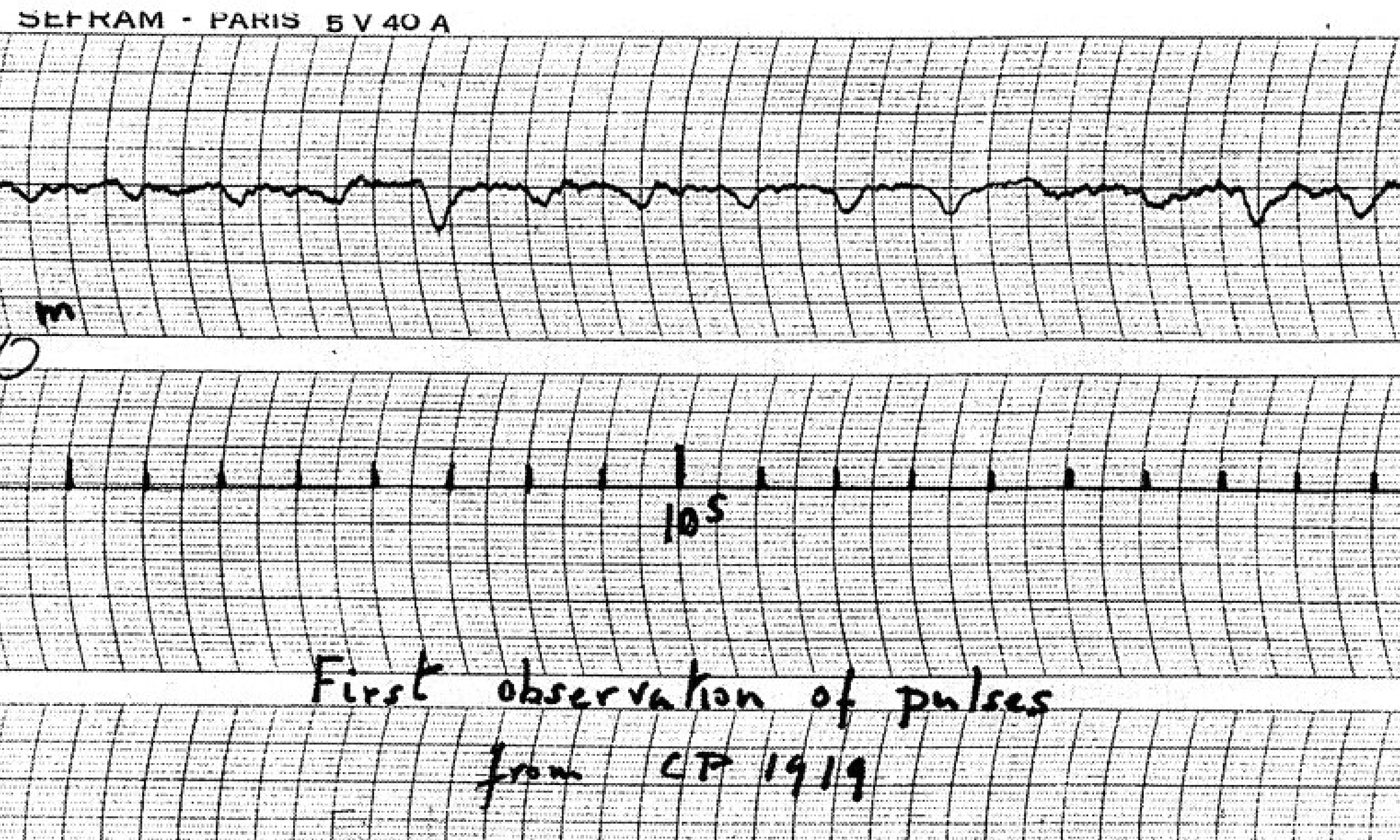
SNR = 8

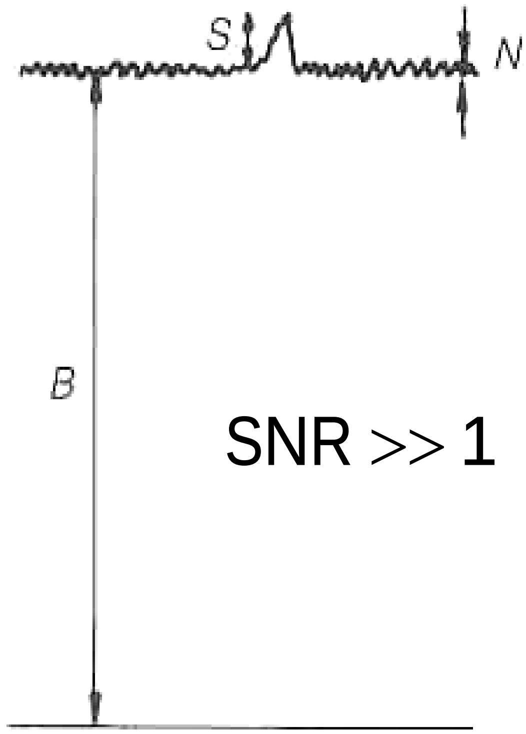
SNR = 16

Pulsars – the 4 Acre Array

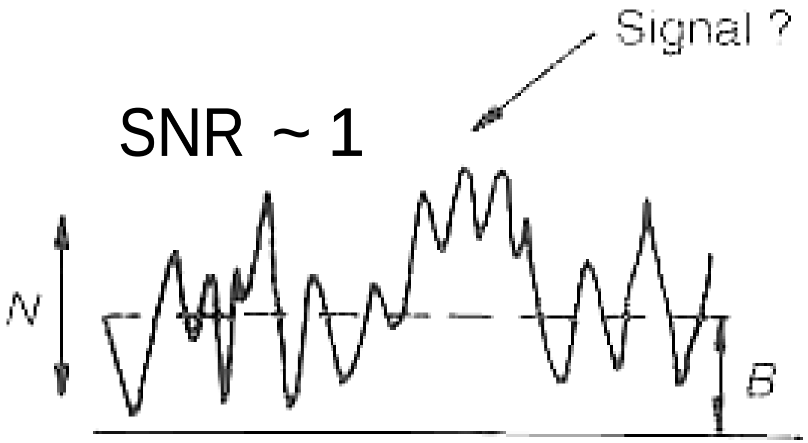
Discovery of pulsars

- First observations of pulses from a pulsar





Here we have a signal, *S*, which is very weak compared to the background, *B* (as might be the case when observing, say, a star or planet at twilight), but is easily detected after the background has been subtracted because the SNR is large (i.e., *S* is large compared with *N*).



Here the signal, the background and the noise have comparable sizes. This makes it difficult to estimate the amount of background to subtract and to know whether there is a signal there at all. The existence of *any* signal is clearly doubtful.

Poisson statistics

- Astronomical observations often involve counting photons. However, the number of photons arriving at our telescope from a given source in a fixed time interval of time will fluctuate.
- We can treat the arrival rate of photons statistically, which means that we can calculate the range of numbers of photons which we expect to arrive in a given time interval.
- We make certain assumptions (axioms):

1. Photons arrive independently in time
2. The average photon arrival rate is a constant

- If our observed photons satisfy these axioms, then the arrival numbers follow a Poisson distribution.

- Suppose the (assumed constant) mean photon arrival rate is R photons per second.

- If we observe for an exposure time τ seconds, then we expect to receive $R\tau$ photons in that time.

- We refer to this as the expectation value of the number of photons, written as

$$\boxed{E(N) = \langle N \rangle = R\,\tau} \tag{4.1}$$

- If we made a series of observations, each of time τ, we wouldn't expect to receive exactly $\langle N \rangle$ photons every time, but the average number of counts should equal

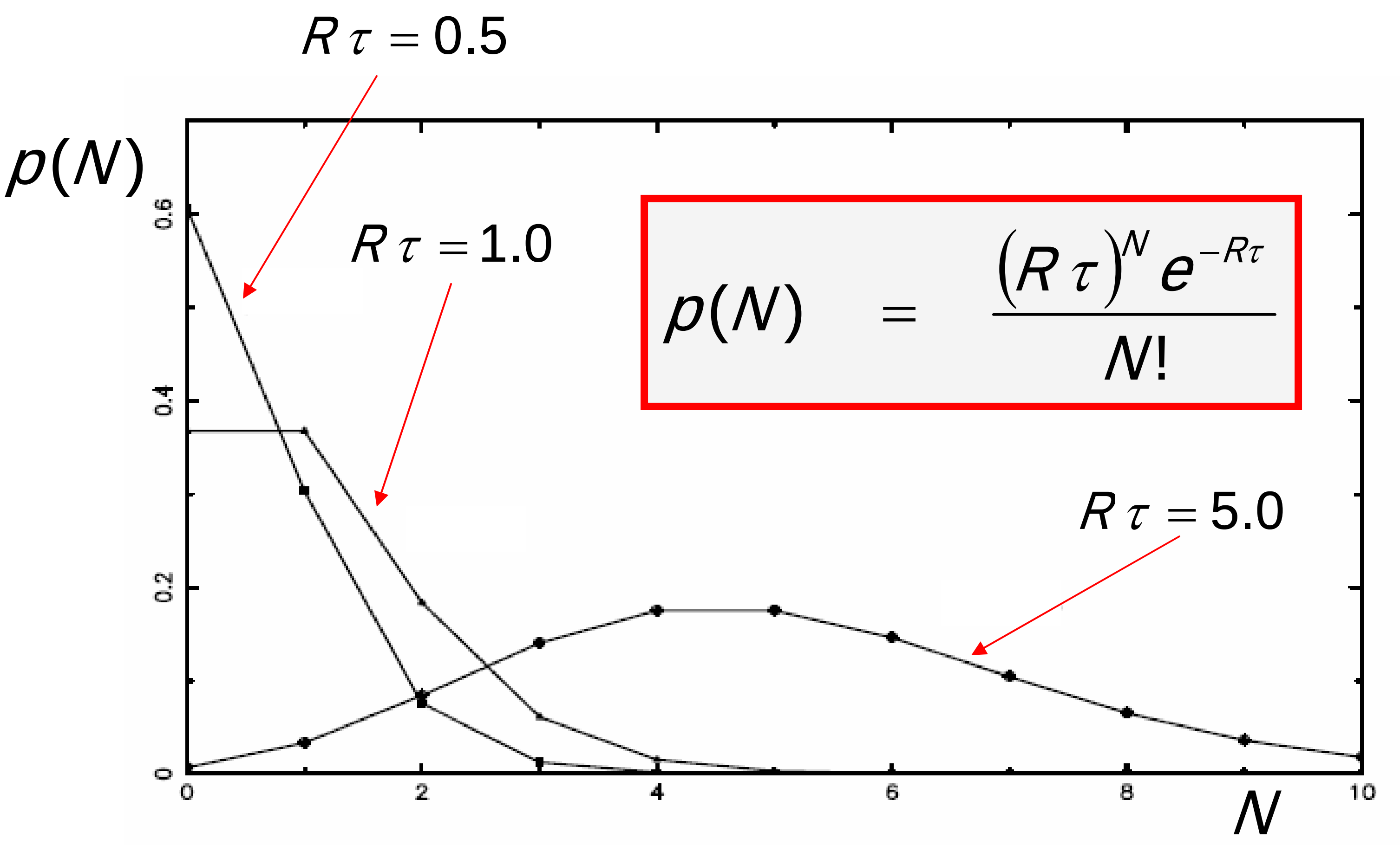
$$\langle N \rangle = R\tau$$

(in fact this is how we can estimate the value of the rate R)

- Given the two Poisson axioms, we can show that the probability of receiving N photons in time τ is given by

$$\boxed{p(N) = \frac{(R\tau)^N e^{-R\tau}}{N!}} \qquad (4.2)$$

Poisson statistics



As $R\tau$ increases, the shape of the Poisson distribution becomes more symmetrical (it tends to a Normal, or Gaussian, distribution).

Poisson statistics

We have already defined the expectation value as $E(N) = R\tau$

We could confirm that this is consistent with the Poisson distribution using eq. (4.2), though we won't (it is!).

We can also define the variance of N, which is a measure of the spread in the distribution:

$$\boxed{\mathrm{var}(N) = \sigma^2 = E\left\{\left[N - E(N)\right]^2\right\}} \qquad (4.3)$$

We can think of the variance as the mean squared-deviation of N from $\langle N \rangle$.

Poisson statistics

For a Poisson distribution, the variance of N can be shown to be

$$\boxed{\mathrm{var}\,(N) = R\,\tau} \tag{4.4}$$

which is the same as the mean. The standard deviation of N is defined as

$$\sigma = \sqrt{\mathrm{var}(N)} = \sqrt{R\,\tau} \tag{4.5}$$

In practice we usually only observe for one period of (say) τ seconds, during which time we receive (say) a count of N_{obs} photons. We can therefore estimate the arrival rate as

$$\boxed{\hat{R} = \frac{N_{\mathrm{obs}}}{\tau}} \tag{4.6}$$

Poisson statistics

We take N_{obs} as our 'best' estimate for $\langle N \rangle \tau$ with error

$$\sigma_N = \sqrt{N_{\text{obs}}}$$

i.e., we quote our experimental estimate for the mean number count of photons in time interval τ as

$$\boxed{N_{\text{obs}} \pm \sqrt{N_{\text{obs}}}} \qquad (4.7)$$

Adding noise

Usually there are several sources of noise in our observation, each with a different variance.

Probability theory tells us that, if the sources of noise are all independent, we can get the total noise variance by adding together the individual variances:

$$\boxed{\sigma^2_{\mathrm{total}} = \sigma^2_{\mathrm{Poisson}} + \sigma^2_{\mathrm{other}}} \qquad (4.8)$$

Sources of Poisson noise:

1) fluctuations in photon count from the sky

2) dark current: thermal fluctuations in a CCD

Noise and telescope / detector design

Suppose we observe a point source, of flux density S_ν, through a telescope with collecting area A for time τ, in bandwidth $\Delta\nu$ centred on ν_0.

The total energy collected by the detector (ignoring any losses) is

$$\boxed{E_{\mathrm{tot}} = S_\nu A \Delta\nu \tau} \tag{4.9}$$

so the number of photons collected is

$$\boxed{N_{\mathrm{tot}} \approx \frac{S_\nu A \Delta\nu \tau}{h\nu_0}} \tag{4.10}$$

Noise and telescope / detector design

Correcting for combined quantum efficiency of telescope and detector

$$\boxed{N_{\text{tot}} = \eta \frac{S_\nu A \Delta\nu \tau}{h\nu_0}} \qquad (4.11)$$

Fraction of incident photons that produce a response in the detector

Thus $\sigma_{\text{Poisson}} = \sqrt{N_{\text{tot}}}$ and $\boxed{\text{SNR} \propto (A \Delta\nu \, \tau)^{1/2}}$

(4.12)

So when the noise is dominated by the counting statistics of photons, the sensitivity of a telescope only increases as the square root of its aperture.

Noise and telescope / detector design

In a radio astronomy the noise is often dominated by noise in the receiver electronics.

$$\text{signal} \propto \text{collected energy} \propto A$$

$$\text{noise is independent of } A$$

$$\text{SNR} \propto A$$

So when the noise is dominated by detector electronics, the sensitivity of the telescope increases in *proportion* to its aperture and the sensitivity of such a telescope goes as

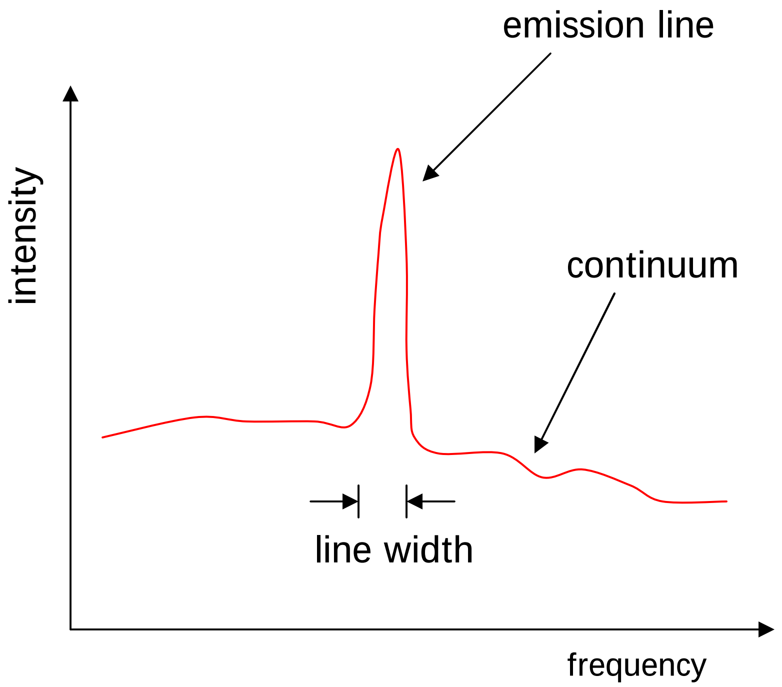
$$\text{SNR} \propto A(\Delta \nu \, \tau)^{1/2}$$

Line sources

Eq. (4.12) suggests that we can increase the signal-to-noise-ratio by increasing the bandwidth of our observation.

This is not necessarily true if we are observing a source which emits only over a narrow frequency range – e.g. a spectral line.

Increasing $\Delta\nu$ beyond the line width will increase the amount of noise (from the background continuum) without further increasing the amount of signal (from the line).



Example

The 2-D image of a faint galaxy observed by a CCD covers 50 pixels. For an exposure of 5 seconds a total of 10^4 photo-electrons are recorded by the CCD from these pixels. An adjacent section of the CCD, covering 2500 pixels, records the background sky count. During the same exposure time a total of 10^5 photo-electrons are recorded from this adjacent section. Show that, after subtracting the background sky count, the estimate of the signal-to-noise ratio for the detecting this galaxy is 73.

Calculate the length of exposure required to increase the signal-to-noise ratio to 100.

Solution:

First we need to calculate the background level, answering the question "how many of the counts in the original 50-pixel image are due to the background?". We are told that 2500 background pixels have generated 10^5 counts. This can be thought of as the sum of fifty 50-pixel patches, so our estimate for the background in one patch is just $10^5/50 = 2000$.

Now for the uncertainty in this: as stated in equation (4.8), if we have several independent contributions to the overall noise we simply add the variances of the contributions to get the total variance. The variance in our estimate of the background over 2500 pixels is simply 10^5 (assuming Poisson statistics), so the variance for each of the patches is, again, $10^5/50=2000$. Our standard deviation (i.e., our uncertainty in the background counts) is therefore the square root of 2000 which is 44.7.

So our estimate for the background count for a 50-pixel patch is

$$N_b = 2000 +/- 44.7$$

Solution (cont):

Our total count (signal + noise) in the galaxy region is 10^4, so we can estimate the number of counts from the galaxy alone as

$$N_g = N_{tot}-N_b = 10000-2000 = 8000.$$

What about the uncertainty in this (i.e., the noise in our measurement)? The variance of N_g is the SUM of the variances of N_{tot} and N_b, i.e.

$$\mathrm{var}[N_g] = \mathrm{var}[N_{tot}] + \mathrm{var}[N_b] = 10000 + 2000 = 12000,$$

and our noise is the square root of this, which is 109.5. So we can say that

$$N_g = 8000 +/- 109.5,$$

and the signal-to-noise ratio is 8000/109.5 = 73 after 5 s of observation.

SNR increases as the square root of the observing time, so to get an SNR of 100 we need to observe for a total of $5x(100/73)^2$ = 9.4 seconds.