Quick facts #2: The two-body problem

Introduction

The most straightforward orbit calculations occur when the central body is much more massive than the orbiting body, as is the case for the orbits of man-made satellites around the Earth. We assumed that this is also the case for planetary orbits about the Sun – a good approximation, especially for the minor planets. However it is *not* appropriate in a binary star system where the masses of the two stars orbiting each other are similar. Even for planetary motion there is a small but important correction to be made once the orbital motion of the Sun is taken into account. The good news is that we can apply all our old results, with suitable modifications.

There are two ways to approach the problem. The first, covered in the notes, is to realise that in general both bodies perform elliptical orbits about a fixed, common, centre of mass. For example, the mass m_1 moves as if in motion about a fixed mass of magnitude $m_2' = m_2^3/(m_1+m_2)^2$. All our 'one-body' results can now be applied, using m_2' fixed at the focus, in place of a moving m_2. Sometimes however, it is useful to think in terms of the *relative* motion of the two masses rather than their individual motions about the centre of mass.

Relative motion

The general elliptical analysis is *just* beyond the scope of this course (though not by much). Instead we will analyse the case of two stars of similar mass performing circular orbits about a common centre (Figure 1).

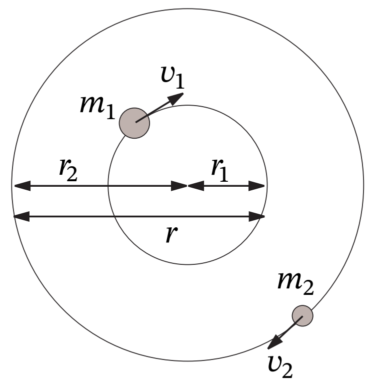

Figure 1: Masses m_1 and m_2 in circular orbits ($m_1 > m_2$).

Let the stars have masses m_1 and m_2, at distances r_1 and r_2 from their common centre. We will define the *radius vector* of the system as

$$r = r_1 + r_2.$$

r is therefore the distance between the stars. It is constant in our example, but would vary if the orbits were elliptical. The two stars must have the same *angular* speed ω (otherwise one mass would catch up the other, and the gravitational force would not be directed towards the centre of the circles). The orbital speeds of the two masses are therefore

$$v_1 = \omega r_1,$$
$$v_2 = \omega r_2.$$

The gravitational force between the stars depends on the inverse-square of their separation, r, and supplies the centripetal force that keeps them moving in circles of radii r_1 and r_2 respectively. So

$$\frac{Gm_1m_2}{r^2} = \frac{m_1v_1^2}{r_1} = m_1\omega^2 r_1 \tag{1}$$
$$= \frac{m_2v_2^2}{r_2} = m_2\omega^2 r_2. \tag{2}$$

The equality of the two right-hand terms gives

$$\frac{r_1}{r_2} = \frac{m_2}{m_1},$$

which is just the condition that the common centre of the circles is the centre of mass of the system. Using Equations 1 and 2 we can also write

$$\omega^2 = \frac{Gm_1}{r^2 r_2} = \frac{Gm_2}{r^2 r_1}. \tag{3}$$

Also, since

$$r = r_1 + r_2 = r_1 + \frac{m_1}{m_2} r_1,$$

we get

$$r_1 = \frac{m_2 r}{m_1 + m_2}.$$

Combining this with Equation 3 we get

$$\omega^2 = \frac{G(m_1 + m_2)}{r^3}.$$

We can write this in terms of the period of the orbit, $T = 2\pi/\omega$, as

$$T^2 = \frac{4\pi^2 r^3}{G(m_1 + m_2)},$$

i.e.,

$$\boxed{\frac{r^3}{T^2} = \frac{G(m_1 + m_2)}{4\pi^2}.}$$

This is Kepler's third law for circular orbits. It is always true, even when one mass is much greater than the other, but is especially useful when the masses are similar. If $m_1 \gg m_2$, then $m_1 + m_2 \simeq m_1$, $r \simeq r_2$ and the equation reduces to the 'one-body' version derived earlier in the course.

In fact all the earlier results can be applied to the two-body problem with the following trick (you will be shown the proof in honours astronomy!). We can define

$$\begin{aligned} M &= m_1 + m_2 \\ \mu &= \frac{m_1 m_2}{m_1 + m_2} \\ v &= v_1 + v_2. \end{aligned}$$

The quantity μ is called the *reduced mass* of the system. Note that if $m_1 \gg m_2$ then $M \simeq m_1$ and $\mu \simeq m_2$.

In general, the two-body problem can be treated as an equivalent one-body problem in which the reduced mass μ is orbiting about a fixed mass M at a distance r.

The reduced mass will describe an imagined ellipse about M with semi-major axis $a = a_1 + a_2$ and eccentricity e (which is the same as the eccentricities of the two real orbits). The radius vector, r, tells us the relative separation of the two masses. Here are some results (derived using the above rule) for the two-body system. Note their similarity to the one-body results derived earlier in the course, but remember the definitions of M, μ, v, a and r in terms of the parameters of the two stars:

Energy:	$E = \frac{1}{2}\mu v^2 - \frac{GM\mu}{r}$
Angular momentum:	$L = \mu r v \sin\theta$
Speed:	$v^2 = GM(2/r - 1/a)$
Period:	$T^2 = \frac{4\pi^2 a^3}{GM}.$

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