The gravitational force from a spherically symmetric body

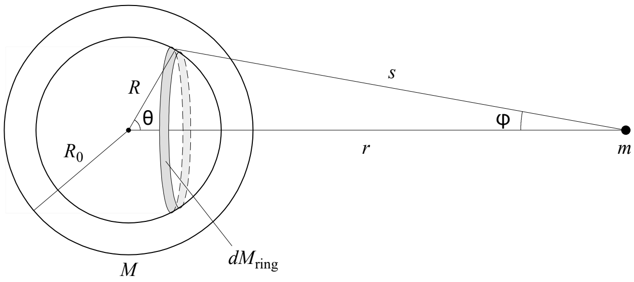

Figure 1: The geometry used in the proof.

This proof is not examinable, and for the enthusiast only! It explains why a spherically symmetric object, such as a star or planet, can be considered as a point mass in gravitational calculations, even when it is very close. Isaac Newton worried about this proof for years, so if it appears difficult at first sight you are in good company. We will use the notation and approach of Carroll & Ostlie *Modern Astrophysics*. The gravitational force exerted by a spherically symmetric object of mass M on a point mass, m, can be calculated by considering the sum of the contributions from rings of matter in M, centred along the line of length r connecting the point mass with the centre of the extended object (Figure 1). All points on a particular ring are the same distance from the point mass, and therefore exert the same force on it. In addition, the resultant force from the ring must be along r, as the mass is symmetric about it. Hence we need only consider the component of gravity from each part of the ring along the line connecting the mass centres.

Let the radius of the extended mass be R_0 and the mass of the ring under consideration be $\mathrm{d}M_{\mathrm{ring}}$. The gravitational force exerted by this ring on the point mass m is

$$\mathrm{d}F_{\mathrm{ring}} = \frac{Gm\,\mathrm{d}M_{\mathrm{ring}}}{s^2}\cos\phi,$$

where s is the distance between the rim of the ring and m, and ϕ the angle it makes with the central line. Assuming that the mass density of the extended object, ρ, is a function of radius only, we can calculate the mass of the ring as

$$\begin{aligned}\mathrm{d}M_{\mathrm{ring}} &= \rho(R)\,\mathrm{d}V_{\mathrm{ring}}\\ &= \rho(R)2\pi R\sin\theta R\,\mathrm{d}\theta\,\mathrm{d}R\\ &= 2\pi R^2\rho(R)\sin\theta\,\mathrm{d}R\,\mathrm{d}\theta.\end{aligned}$$

From the diagram we can make the substitution

$$\cos\phi = \frac{r - R\cos\theta}{s}.$$

By the cosine rule, s is

$$s = (r^2 - 2rR\cos\theta + R^2)^{1/2}.$$

Substituting into the expression for $\mathrm{d}F_{\mathrm{ring}}$, we must now sum over all rings in a shell (i.e., integrate θ from 0 to π) and over all shells in the sphere (i.e., integrate R from 0 to R_0) to get the total gravitational force:

$$\begin{aligned}F &= Gm\int_0^{R_0}\int_0^{\pi}\frac{(r - R\cos\theta)\rho(R)2\pi R^2\sin\theta}{s^3}\,\mathrm{d}\theta\,\mathrm{d}R\\ &= 2\pi Gm\int_0^{R_0}\int_0^{\pi}\frac{rR^2\rho(R)\sin\theta}{(r^2+R^2-2rR\cos\theta)^{3/2}}\,\mathrm{d}\theta\,\mathrm{d}R\\ &\quad - 2\pi Gm\int_0^{R_0}\int_0^{\pi}\frac{R^3\rho(R)\sin\theta\cos\theta}{(r^2+R^2-2rR\cos\theta)^{3/2}}\,\mathrm{d}\theta\,\mathrm{d}R.\end{aligned}$$

The integration over θ can be carried out by making the substitution $u = s^2 = r^2 + R^2 - 2rR\cos\theta$, so that $\sin\theta\,\mathrm{d}\theta = \mathrm{d}u/(2rR)$. After integrating over u, the force equation now becomes

$$F = \frac{Gm}{r^2}\int_0^{R_0} 4\pi R^2\rho(R)\,\mathrm{d}R.$$

The integrand of this expression is simply the mass, $\mathrm{d}M_{\mathrm{shell}}$, of a single shell of matter, thickness $\mathrm{d}R$ and volume $4\pi R^2\,\mathrm{d}R$, giving a force contribution of

$$\mathrm{d}F_{\mathrm{shell}} = \frac{Gm\,\mathrm{d}M_{\mathrm{shell}}}{r^2}.$$

i.e., *the gravitational attraction from the shell is as if all its mass was concentrated at its centre.* The total expression for F is just the sum of these:

$$F = G\frac{Mm}{r^2},$$

so the attraction of the whole sphere is as if all its mass were concentrated at its centre.

GW
2017
v.464